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Smooth Actions of Compact Lie Groups on S^2 are Smoothly Equivalent to Linear Actions

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1. Introduction

Mathematicians have been interested in group actions on spheres since before the algebraic description of a group was defined. The rotational and reflective symmetries of the circle and of S^2 were naturally among the first to be considered. When we restrict attention to a compact topological group, there is a classic theorem of Kerékjártó to the effect that for S^2, these are essentially the only actions:

Theorem 1 (Kerékjártó, [3])**.** *Every continuous, effective action of a compact topological group G on S^2 is topologically conjugate to a linear action (to the standard action of a subgroup of O_3 on S^2 as a subset of $\mathbb{R}^3$).*

Thus in the topological category, in order to understand all effective actions of compact groups on S^2, it is enough to understand the subgroups $G \leq O_3$ and their actions via matrix multiplication on $S^2 \subseteq \mathbb{R}^3$ (*henceforth referred to as linear actions*). The higher dimensional analogues of this theorem are decidedly false, as shown by Bing [1] and others, so Kerékjártó's result can be seen as a statement about how restrictive low dimensional actions can be.

Whenever there are two actions $\alpha, \lambda : G \times X \to X$, the phrase *topologically conjugate* in the above theorem simply means that there is a homeomorphism $f : X \to X$ that for all $g \in G$ satisfies $f(\alpha(g, x)) = \lambda(g, f(x))$, *i.e.* the following diagram commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ {\scriptstyle \mathrm{id}\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times X & \xrightarrow[\lambda]{} & X \end{array}$$

In other words there is an equivariant homeomorphism between the corresponding G-spaces (X, α) and (X, λ). We will refer to an equivariant map between two G-spaces with the same underlying set as a *conjugating function (from α to λ)*.

For effective actions, the existence of the conjugating function f tells us important algebraic information about the structure of G, but the fact that the map f is continuous is an added benefit that should not be overlooked. If we were to take G to be a compact Lie group, with a smooth action, it would be natural to expect then that f could be assumed smooth as well. This turns out to be true, and is the content of our main theorem:

Theorem 2. *Every smooth, effective action of a compact Lie group on S^2 is smoothly conjugate to a linear action.*

In other words, given these extra assumptions on the group action, we may take the conjugating homeomorphism supplied by Theorem 1 to be a diffeomorphism. This result was stated as fact in the survey article [14] with reference to a forthcoming (at the time) paper of Edmonds [5]. Actually Edmonds makes no statement to this extent, and we have found no further references in the literature. Certainly such an extension is a desirable result, and despite having been assumed to be true, deserves to be written down explicitly. To justify this claim, we offer here an interesting corollary of Theorem 2 for the transitive case:

Corollary 3 (Essentially due to Palais, [12])**.** *Every continuous, homomorphic embedding of SO_3 into Diff(S^2) is the result of conjugating the standard inclusion λ by a unique orientation preserving diffeomorphism.*

In Section 6 We will show that this corollary follows easily from Theorem 2, together with a powerful theorem of Palais [12, Cor. 2] regarding the topology of spaces of smooth actions.

The current paper was inspired by a more recent article of Kolev [8], wherein Theorem 1 is reproven using modern techniques. Some of the topological arguments that carry over to the smooth category are borrowed from this article, and I am indebted to him for writing it. Thanks are also due to Heiner Dovermann for his ideas, advice and patience.

2. Preliminaries

2.1. **Group Actions.** The material for this section can be found in Bredon [2, p.32]. A G-set (X, α) is a set X together with a function $\alpha : G \times X \to X$, subject to the following conditions:

$$\alpha(e,x) = x \quad \forall x \in X\,, \;\text{ and}$$
$$\alpha\big(g, \alpha(h,x)\big) = \alpha(gh,x) \quad \forall g,h \in G \;\;\&\;\; x \in X.$$

These conditions guarantee that the maps $\alpha(g,-) : X \to X$ are each bijections, and that $\alpha(g^{-1},-)$ is the inverse of $\alpha(g,-)$. We will employ the standard notation for these maps $\alpha_g(x) := \alpha(g,x)$. With this convention, we can write the properties above succinctly as:

$$\alpha_e = \mathrm{id}_X\,, \quad \alpha_g \circ \alpha_h = \alpha_{gh}\,, \quad \alpha_{g^{-1}} = \alpha_g^{-1}$$

The map α is referred to as the *action* (of G on X). Occasionally, when the action is well understood, a G-set (X,α) will be referred to simply as X. We will have occasion to consider multiple actions α and λ on the same set X, giving rise to the G-sets (X,α) and (X,λ). Whenever the underlying set is understood, it is often useful to refer to the actions themselves, without mention of X.

If (X,α) and (Y,β) are two G-sets, a map $f : X \to Y$ is said to be G-*equivariant* if the following diagram commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{\;\alpha\;} & X \\ {\scriptstyle \mathrm{id}\times f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ G \times Y & \xrightarrow[\;\beta\;]{} & Y \end{array}$$

If the group G is a topological group, and X is a topological space, it makes sense to consider continuous actions. Similarly if G is a Lie group, and X is a smooth manifold, it makes sense to consider smooth actions and we specialize our discussion to this case from now on.

For any smooth manifold X, the set $\mathrm{Diff}(X)$ is defined to be the set of all diffeomorphisms of X. This set has a natural group structure with composition of maps as multiplication. For every smooth group action α of a Lie group G on X, there is an associated homomorphism $\tilde{\alpha}$ defined by

$$\tilde{\alpha} : G \to \mathrm{Diff}(X)\,, \quad \tilde{\alpha} : g \mapsto \alpha_g$$

If α has the property that for every $g \in G$, there is some $x \in X$ such that $\alpha_g(x) \neq x$, we say that the action is *effective*. This condition is equivalent to the statement that $\alpha_g = \mathrm{id}_X$ if and only if $g = e$. Said another way, and action is effective precisely when $\ker(\tilde{\alpha}) = \{e\}$. The First Isomorphism Theorem implies that $\tilde{\alpha}(G) \cong {}^{G}\!/_{\ker(\tilde{\alpha})}$, and so an effective action is one whose associated homomorphism $\tilde{\alpha}$ is a homomorphic embedding of $G \hookrightarrow \mathrm{Diff}(X)$. In some sense, this shows that the only 'interesting' actions are effective, and in light of this we will only consider effective actions in this paper.

Given a group action α of G on X, there are particular subsets of both G and X that yield important information about α. For an element $g \in G$, the set $\mathrm{Fix}_\alpha(g) := \{x \in X \mid \alpha_g(x) = x\}$ is called the *fixed point set* (of g). The set

$$\mathrm{Fix}_\alpha(G) := \bigcap_{g \in G} \mathrm{Fix}_\alpha(g)$$

is the set of *fixed points of the action* α.

Dually, for an element $x \in X$, the set $\mathrm{Stab}_\alpha(x) := \{g \in G \mid \alpha_g(x) = x\}$ is called the *stabilizer* of x (under α). The set $\mathrm{Stab}_\alpha(x)$ is easily seen to be a subgroup of G for any $x \in X$, and if G is a topological group, $\mathrm{Stab}_\alpha(x)$ is a closed subgroup. For an element $x \in X$, the set $\mathrm{Orb}_\alpha(x) := \{\alpha_g(x) \in X \mid g \in G\}$ is called the *orbit* of x (under α).

For any element $y \in \mathrm{Orb}_\alpha(x)$, there is some $g \in G$ such that $gx = y$. If $h \in \mathrm{Stab}_\alpha(x)$, then $ghg^{-1}(y) = gh(x) = g(x) = y$, and so $ghg^{-1} \in \mathrm{Stab}_\alpha(y)$. This shows that the elements of a given orbit have stabilizers that are conjugate to one another, *i.e.* $\mathrm{Stab}_\alpha(gx) = g\mathrm{Stab}_\alpha(x)g^{-1}$. The collection $\{\mathrm{Orb}_\alpha(x) \mid x \in X\}$ of all orbits of elements in X is called the quotient space (of the G-set (X, α)) and is denoted by ${}^{X}\!/_{G}$. The relevant theorem that relates orbits and stabilizers is the following:

Theorem 4 (Smooth Orbit/Stabilizer Theorem)**.** *If α is a smooth action of a compact Lie group G on a compact manifold X, then*

$${}^{G}\!/_{Stab_\alpha(x)} \cong Orb_\alpha(x)$$

where '≅' means diffeomorphism, and the coset space has the canonical induced smooth structure.

For more details and a proof of this theorem see Bredon [2, Cor. 1.3, p.303]. The above has as a special case, a more classical version of the Orbit/Stabilizer Theorem, which says that $|G| = |\mathrm{Stab}_\alpha(x)| \cdot |\mathrm{Orb}_\alpha(x)|$ whenever G and X are both finite.

2.2. **Perspective on Groups.** To prove Theorem 2, we would like to replace the conjugating homeomorphism f supplied by Theorem 1 with an equivariant diffeomorphism $\mathfrak{f}$.

Important Note: *Throughout this paper, $\mathfrak{f}$ will refer to the equivariant diffeomorphism we are attempting to construct.*

During the proof of Theorem 2, we expect to consider all smooth, effective actions α of a compact Lie group G on S^2. The conjugating function f will allow us to associate a matrix group $F_f(G)$ to the given group G. Our case by case proof of Theorem 2 will be indexed by preferred representatives of the possible conjugacy classes of $F_f(G)$ within the matrix group O_3. This section establishes these ideas.

The set, $O_3 := \{A \in M_3(\mathbb{R}) \mid A^t = A^{-1}\}$ forms a group under matrix multiplication. The group O_3 has a natural action $\mu : O_3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by multiplication of a vector by a matrix: $(A, v) \mapsto Av$. If $(A, v) \in O_3 \times S^2$, then

$$\|Av\| = \sqrt{(Av)^t(Av)} = \sqrt{v^tA^tAv} = \sqrt{v^tv} = \|v\| = 1,$$

Thus the action μ restricts to a map $\Lambda := \mu \mid_{O_3\times S^2}$. The map Λ is a smooth action of O_3 on S^2, and will be referred to as the *standard action.* The associated homomorphism $\tilde{\Lambda} : O_3 \hookrightarrow \mathrm{Diff}(S^2)$ will be referred to as the *canonical inclusion.*

Now suppose α is a smooth action of a compact Lie group G on S^2. By Theorem 1, there is a homeomorphism $f : S^2 \to S^2$, such that $f \circ \alpha_g \circ f^{-1}$ is a linear transformation for all $g \in G$. Let $F_f(g)$ be the matrix corresponding to $f \circ \alpha_g \circ f^{-1}$ in the standard basis for $\mathbb{R}^3$. If we set $\lambda(g, x) = f \circ \alpha_g \circ f^{-1}(x) = F(g) \cdot x$, this defines a linear action $\lambda : G \times S^2 \to S^2$.

We have the following relationships between α, λ and Λ:

$$f\big(\alpha(g, f^{-1}(x))\big) = \lambda(g, x) = \Lambda(F_f(g), x)$$

$$f \circ \alpha_g \circ f^{-1} = \lambda_g = \Lambda_{F(g)}$$

$$\tilde{\lambda} = \tilde{\Lambda} \circ F_f$$

Once we have found the linear action λ, conjugation by an element $h \in O_3$ gives rise to another linear action λ^h of G, defined by $\lambda^h_g = h\lambda_g h^{-1}$ for each $g \in G$. This yields

$$\lambda_g \circ f = f \circ \alpha_g$$

$$\Longleftrightarrow$$

$$h^{-1}\lambda^h_g h \circ f = f \circ \alpha_g$$

$$\Longleftrightarrow$$

$$\lambda^h_g \circ (h \circ f) = (h \circ f) \circ \alpha_g$$

$$\Longrightarrow$$

$$F_{h\circ f}(g) = h \cdot F_f(g) \cdot h^{-1}$$

$$\Longrightarrow$$

$$F_{h\circ f}(G) = h \cdot F_f(G) \cdot h^{-1}.$$

Our goal is to find a smooth conjugating function $\mathfrak{f}$ from α to the a linear action λ. Since λ^h is also a linear action, we could just as easily find a smooth conjugating function from α to λ^h and this would also prove Theorem 2 for the action α. In fact, $h \circ f$ is smooth if and only if f is smooth, so we are free to choose whichever $h \in O_3$ makes λ^h particularly nice. Now λ^h is the standard action of $h \cdot F_f(G) \cdot h^{-1}$, so we know that every smooth action α of a compact Lie group on S^2 is topologically conjugate to the standard action of some preferred representative of the conjugacy class determined by $F_f(G)$.

For $K \leq O_3$ define the conjugacy class $[K] := \{hKh^{-1} \mid h \in O_3\}$, and the set $C_c(O_3) := \{[K] \mid K \leq O_3 \text{ is a compact, Lie subgroup}\}$ of all conjugacy classes of compact Lie subgroups.

In Section 3, we will determine a system of distinct representatives[1] of $C_c(O_3)$ that will serve as the indexing set for the cases of the proof of Theorem 2.

2.3. **G-CW Complexes.** In the case where $[F_f(G)] \in C_c(O_3)$ is the conjugacy class of a finite subgroup of O_3, the proof of Theorem 2 proceeds by building up a new, smooth map $\mathfrak{f}$ piece by piece. This is done by defining $\mathfrak{f}_0$ to be f restricted to a finite subset X_0 of S^2, then taking successive extensions of $\mathfrak{f}_0$ to larger and larger subsets. In order to carry out such a construction, it is necessary to have an increasing sequence of subsets $X_0 \subset X_1 \subset X_2 = S^2$ that are invariant under the action λ of G corresponding to our chosen representative $F_f(G) \leq O_3$. The structure we will need is known as a finite G-CW complex, and is an equivariant version of the more well known finite CW complex which we define presently.

Let X be a topological space. A subset of $C \subseteq X$ is called an *n-cell* if it is homeomorphic to the open unit disk $D^n := \{v \in \mathbb{R}^n \mid \|v\| < 1\}$, and is called a *cell* if it is an n-cell for some $n \in \mathbb{N}$. For a cell C, we define $\dim(C) = n$ if C is an n-cell. By L.E.J. Brouwer's Theorem on Invariance of Domain, dim is well-defined.

A finite CW complex (X, P) is a Hausdorff space X together with a partition P of X into finitely many cells, subject to the following criteria[2]:

- For every $C \in P$ with $\dim(C) = n$, there is a continuous map ϕ_C from the closed unit disk $\overline{D^n} := \{v \in R^n \mid \|v\| \leq 1\}$ to X with the following two properties:
  - (i) The image $\phi_C(S^{n-1})$ is contained in a union of cells of P whose dimensions are strictly less than n.
  - (ii) The restriction of ϕ_C to D^n is a homeomorphism onto its image $\text{im}(\phi_C\big|_{D^n}) = C \subseteq X$.

Now we will modify this construction so that it nicely incorporates a group action.

[1]These will be the '*nice*' preferred representatives that we referred to earlier.

[2]This definition is only valid for *finite* CW complexes. The closure-finite and weak topology axioms for a general CW complex are automatically satisfied for finite complexes.

Let (X, λ) be a G-set where X is a topological space, and λ is a continuous action[3]. An *equivariant n-cell* is a subset $C \subseteq X$ that is equivariantly homeomorphic to ${}^{G}\!/_{H_C} \times D^n$, where G acts by left multiplication on the left factor and trivially on the right factor, for some closed subgroup $H_C \leq G$. A subset C is called an *equivariant cell* if it is an equivariant n-cell for some n. Define $\dim(C) = n$ if C is an equivariant n-cell.

A finite G-CW complex (X, λ, P) is a triple where (X, λ) is a Hausdorff G-space, and P is a partition of X into equivariant cells, and the following conditions hold:

- For every $C \in P$ with $\dim(C) = n$, there is a continuous map ϕ_C from ${}^{G}\!/_{H_C} \times \overline{D^n}$ to X with the following two properties:
  (i) The image $\phi_C\left({}^{G}\!/_{H_C} \times S^{n-1}\right)$ is contained in a union of cells of P whose dimensions are strictly less than n.
  (ii) The restriction of ϕ_C to ${}^{G}\!/_{H_C} \times D^n$ is an equivariant homeomorphism onto C.

In either the plain or equivariant case, the map ϕ_C is called a *characteristic map* for the cell C. If (X, P) is a finite CW complex or (X, λ, P) is a finite G-CW complex, define the collection $P_k := \{C \in P \mid \dim(C) \leq k\}$. The k-skeleton of X is then defined to be the subset:

$$X_k := \bigcup_{\dim(C) \leq k} C$$

The definition of an equivariant cell ensures that the quotient space ${}^{X}\!/_{G}$ inherits the structure of a CW complex when it is endowed with the quotient topology from X. If (X, λ, P) is a G-CW complex and $c \in X$, then $c \in C$ for a unique $C \in P$. Thus by definition of an equivariant cell, there is map ϕ_C and a corresponding $(gH_C, x) \in {}^{G}\!/_{H_C} \times D^n$. By the Orbit/Stabilizer Theorem, ${}^{G}\!/_{H_C} \cong \mathrm{Orb}_\lambda(c) \in {}^{X}\!/_{G}$. This shows how the equivariant cell C corresponds to a unique (non-equivariant) cell in the quotient space.

In Section 4 we will construct explicit decompositions of S^2 into equivariant cells for the standard action of the chosen representatives discussed in Section 3.1.

[3]In other words, (X, λ) is a G-space.

2.4. **Facts About the Groups O_3 and SO_3.** Here we establish some important facts about O_3 and SO_3. Firstly there is a homomorphic projection $p : O_3 \to SO_3$ that we will need in order to classify the elements of $C_c(O_3)$. Secondly every element of SO_3 is either the identity matrix I, or a rotation about some axis through the origin in $\mathbb{R}^3$. Finally we quote a classification theorem for finite subgroups of SO_3, and strengthen it to meet our needs for Theorem 2.

For the first fact, note that the determinant defines a homomorphism $\det : O_3 \to \{\pm 1\}$. The kernel of det is the set of elements of O_3 that preserve orientation, otherwise known as SO_3. The center of O_3 is the subgroup $\{\pm I\}$. The element $-I$ has determinant $\det(-I) = (-1)^3 = -1$, so $-I \notin SO_3$. Let $a \in O_3$, and assume standard matrix notation of $-a := -I \cdot a$. If $\det(a) = -1$, then $\det(-a) = \det(-I \cdot a) = (-1)(-1) = 1$ and we find that $-a \in SO_3$. Define a map

$$p : O_3 \to SO_3\,, \quad p : a \mapsto \det(a) \cdot a$$

Evidently p is a homomorphism, and what's more, it is a surjection that is split by the subset inclusion map : $\iota : SO_3 \to O_3$. Because of this, it is easy to see that $O_3 \cong \{\pm 1\} \times SO_3$ $(a = (\det(a), \det(a)a))$, and that under this isomorphism p corresponds to projection onto the second factor. Knowing that p is a projection is helpful for analyzing O_3, but thinking of p as a *geometric* projection is often misleading. Here is an example:

$$p\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = (-1)\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

This shows that applying p to reflection across the yz-plane produces rotation by π radians about the x-axis.

Let us examine how knowledge of the isomorphism $O_3 \cong \{\pm I\} \times SO_3$ allows us to distinguish between conjugacy classes. Let $H, K \leq O_3$.

Since p is a homomorphism, $[H] = [K] \in C_c(O_3)$ implies $[p(H)] = [p(K)] = C_c(SO_3)$. The contrapositive is

Fact 1. $[p(H)] \neq [p(K)]$ *implies* $[H] \neq [K]$.

Suppose that $-I \in H$ and $-I \notin K$. Since $-I$ is central in O_3, $-I \in gHg^{-1}$ for all $g \in O_3$ and so $[H] \neq [K]$. For reference, we have:

Fact 2. *$-I \in H$ and $-I \notin K$ implies $[H] \neq [K]$*

The kernel of the determinant map restricted to H is $H \cap SO_3$. This kernel must be preserved under conjugation, and so another tool will be:

Fact 3. *$[H \cap SO_3] \neq [K \cap SO_3]$ implies $[H] \neq [K]$.*

The map $p : O_3 \to SO_3$ is a two to one covering map. If $-I \in H$, then $\pm h \in H$ for every $h \in H$ and since $-I$ is central in O_3, $-I$ commutes with every element of H. This gives is our fourth and final tool:

Fact 4. *If $-I \in H$, then $H \cong \{\pm I\} \times p(H)$*

Now we examine SO_3. Note that every element of $a \in SO_3$ has a 1 as an eigenvalue, for consider $\det(a - I)$:

$$\det(a - I) = \det(a - aa^t) = \det(a)\det(I - a^t) = (-1)^3 \det(a - I)$$

$$\Longrightarrow$$

$$\det(a - I) = 0.$$

Now set e_1 to be a unit vector in the 1-eigenspace of a. Extend the set $\{e_1\}$ to an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. With this basis, the condition $a^t = a^{-1}$ implies that a has the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x & y \\ 0 & -y & x \end{bmatrix},$$

where $1 = \det(a) = x^2 + y^2$. This shows that a is a rotation (possibly by 0 radians) about the axis determined by e_1. Let $\theta \in \mathbb{R}$ be such that $x = \cos(\theta)$ and $y = \sin(\theta)$, and with minimal absolute value among such angles. For the limiting case where θ could be π or $-\pi$ we make the arbitrary choice $\theta = \pi$.

Define $a^{1/2}$ to be the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta/2) & \sin(\theta/2) \\ 0 & -\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

Then we have that $a^{1/2} \in SO_3$, and $(a^{1/2})^2 = a \in SO_3$. By repeating this argument, we can construct $a^{1/4}$, $a^{1/8}$ etc.

Now suppose $H \leq SO_3$ is a closed Lie subgroup of $\dim(H) \geq 1$. We will show that H must contain a subgroup isomorphic to S^1. If $a \in H$, then a must be a rotation about some axis in $\mathbb{R}^3$. If a corresponds to a rotation by a rational multiple of π, then a has finite order. Since $\dim(H) \geq 1$, H must contain elements of infinite order, and these necessarily correspond to rotations by irrational multiples of π. Let c be such an element and let ℓ be its axis of rotation. The subgroup $\langle c \rangle \leq H$ is dense in the subgroup of all rotations about ℓ, so the closure $K = \overline{\langle c \rangle}$ is isomorphic to the circle group S^1. Since H is closed, $K \leq H$.

To prove Theorem 2, there will come a point when we need to know the following fact:

Theorem 5. *Every finite subgroup of O_2 is either cyclic, or dihedral. The group $\mathbb{Z}_2$ has two distinct conjugacy classes: one that preserves orientation, and one that does not; and any other subgroups are conjugate within O_2 if and only if they are isomorphic.*

We will not prove Theorem 5 here, but rather suggest that the interested reader refer to Rees [13, Thm 9, p.22] for the classification up to isomorphism, and attempt to prove the conjugacy classification themselves as it is not difficult. We will need the analogue of this for dimension three as well, but it will not come so easily. We begin working our way towards this three dimensional version with a well known result. In Wolf's classical text 'Spaces of Constant Curvature'[17], he proves a classification result for finite subgroups of SO_3:

Theorem 6 (Wolf, Thm 2.6.5, p.85)**.** *Every finite subgroup of SO_3 is either cyclic, dihedral, tetrahedral, octahedral or icosahedral. Furthermore, any two such subgroups are conjugate within SO_3 if and only if they are isomorphic.*

Using the decomposition provided by the projection p, we know that $a = \det(a)p(a)$. Conjugation by an element in O_3 gives

$$aHa^{-1} = \det(a)p(a)Hp(a)^{-1}\det(a) = p(a)Hp(a)^{-1}.$$

This shows that conjugation of some $H \leq SO_3 \leq O_3$ by elements in O_3 is equivalent to conjugation within SO_3. Thus we obtain an easy extension of Theorem 6:

Corollary 7. *Every finite subgroup of SO_3 is either cyclic, dihedral, tetrahedral, octahedral or icosahedral. Furthermore, any two such subgroups are conjugate* ***within*** *O_3 if and only if they are isomorphic.*

We will need a similar classification result for the case of compact Lie subgroups of SO_3 up to conjugacy, and we present a direct argument for this classification here.

Proposition 8. *If $H \leq SO_3$ is closed, and* $\dim(H) \geq 1$, *then H is conjugate to either S^1, $S^1 \rtimes \mathbb{Z}_2$ or SO_3.*

Proof. Let $H \leq SO_3$ with $\dim(H) \geq 1$. By our discussion above, there is some $K \leq H$ with $K \cong S^1$. Let ℓ_z be the z-axis, and let ℓ be the axis of rotation of (all elements of) K. Let $\rho \in SO_3$ be any rotation that maps ℓ_z onto ℓ. If $K = H$, then H is conjugate via ρ to the embedding of S^1 as rotations about ℓ_z. Any other embeddings H' of S^1 into SO_3 are completely determined by their 1-eigenspace ℓ'. Since these are conjugate to rotation about ℓ_z, they are all conjugate to one another.

Whenever we need to refer to the image $\Lambda_g(Y)$ of a subset $Y \subseteq \mathbb{R}^3$ under the linear transformation Λ_g, we will use the notation $g \cdot Y$.

If $K \subsetneq H$, there is some $g \in H \setminus K \subseteq SO_3$. This element g must also have a nontrivial 1-eigenspace, say L. Notice that $L \neq \ell$, for otherwise this would imply that $g \in K$. There are two possibilities: $g \cdot \ell \neq \ell$, or $g \cdot \ell = \ell$.

Case 1: There is some $g \in H \setminus K$ such that $g \cdot \ell \neq \ell$. If such a g exists, then the points $\{\pm u\} = L \cap S^2$ have as stabilizer the subgroup $gKg^{-1} \cong S^1$. Thus H acts with two distinct infinite stabilizers. This guarantees that H acts transitively on S^2, and hence that

$H = SO_3$ (see for example Kolev [8, Lemma 6.3, p.210]). The only subgroup of SO_3 that acts transitively on S^2 is SO_3 itself, so $H = SO_3$.

Case 2: For every $g \in H \setminus K$, $g \cdot \ell = \ell$. If this is the case, then since $L \neq \ell$, g must swap the fixed points $\pm v \in \ell \cap S^2$ of K. This implies that g corresponds to rotation of π radians about the line L, and $L \subseteq \ell^\perp$. This further implies that H contains the elements $g\theta$ for every $\theta \in K$, and every element of this form is a rotation of π radians about some line $L' \subseteq \ell^\perp$. It is a simple matrix calculation to see that $g\theta g = \theta^{-1}$, and hence $H \cong S^1 \rtimes \mathbb{Z}_2$. All of the homomorphic embeddings of $S^1 \rtimes \mathbb{Z}_2$ inside of SO_3 are completely determined by the 1-eigenspace ℓ of the S^1 subgroup, and are hence conjugate to one another by a rotation.

□

2.5. **Riemannian Geometry.** Here we assume the reader has familiarity with the notions of smooth manifolds and their tangent bundles. A good introduction to this topic, as well as the finer points of Riemannian Geometry is do Carmo's book [4], and we assume a working understanding of chapters 0 and 1 from this text.

A Riemannian metric on a smooth manifold M is an assignment of an inner product $\langle\ ,\ \rangle_p$ to every tangent space T_pM, in such a way that if $\mathcal{X}$ and $\mathcal{Y}$ are any smooth vector fields on M, then the map $m_{\mathcal{X},\mathcal{Y}} : M \to \mathbb{R}$ defined by

$$m_{\mathcal{X},\mathcal{Y}} : p \longmapsto \langle \mathcal{X}_p, \mathcal{Y}_p \rangle_p$$

is smooth. A pair $\big(M, \langle\ ,\ \rangle\big)$ of smooth manifold together with a Riemannian metric is called a Riemannian manifold. By a partition of unity argument[4], it can be shown that every (paracompact) smooth manifold admits a Riemannian metric. If α is a smooth action of G on M, then the metric is said to be α-invariant if for any $u, v \in T_pM$,

$$\langle d\alpha_g u, d\alpha_g v \rangle_{\alpha_g(p)} = \langle u, v \rangle_p$$

for every $g \in G$.

Here we introduce the Haar integral following the presentation of Bredon in [2, p.11], and use it to prove the existence of α-invariant Riemannian metrics.

[4]This is given as problem 2-C in [9, p.23]

To every Lie group G, one can associate the vector space $\mathcal{C}^\infty(G,\mathbb{R})$ of smooth maps from G to $\mathbb{R}$. This vector space inherits a partial ordering from the partial ordering $\leq$ on $\mathbb{R}$: $\phi \leq \psi$ if and only if $\phi(g) \leq \psi(g)$ for all $g \in G$. For any $r \in \mathbb{R}$, let $\underline{r} : g \mapsto r$ denote the constant function sending all of G to r.

Let $\phi \in \mathcal{C}^\infty(G,\mathbb{R})$, and $g, h \in G$. Define the linear operator $R_g : \mathcal{C}^\infty(G,\mathbb{R}) \to \mathcal{C}^\infty(G,\mathbb{R})$ by $R_g\phi(h) := \phi(hg)$. If G is compact, then there exists a unique function $\mathcal{I} : \mathcal{C}^\infty(G,\mathbb{R}) \to \mathbb{R}$ called the Haar integral, that is $\mathbb{R}$-linear, monotonic and satisfies:

- (Normalized) $\mathcal{I}(\underline{1}) = 1$
- (G-invariant) $\mathcal{I}(R_g\phi) = \mathcal{I}(\phi)$ for all $\phi \in \mathcal{C}(G,\mathbb{R})$ and $g \in G$.

Using this, we will create a new α-invariant metric from the old one. Let α be a smooth action of G on M, let $p \in M$ and $u, v \in T_pM$. Define a function $\phi_{u,v}(h) := \langle d\alpha_h u, d\alpha_h v\rangle_{\alpha_h(p)}$. The function $\phi_{u,v}(h)$ essentially uses the metric $\langle\ ,\ \rangle$ to measure how the diffeomorphism α_h affects the tangent vectors u and v. Let us verify a way in which two of our notations relate:

$$\begin{aligned}
\phi_{d\alpha_g u, d\alpha_g v}(h) &= \big\langle d\alpha_h(d\alpha_g u), d\alpha_h(d\alpha_g v)\big\rangle_{\alpha_h(\alpha_g(p))} \\
&= \langle d\alpha_{hg} u, d\alpha_{hg} v\rangle_{\alpha_{hg}(p)} \\
&= \phi_{u,v}(hg) \ = \ R_g\phi_{u,v}(h) \\
&\Longrightarrow \\
\phi_{d\alpha_g u, d\alpha_g v} &= R_g\phi_{u,v}.
\end{aligned}$$

Using $\phi_{u,v}$, we define a new Riemannian metric $\langle\ ,\ \rangle^\alpha$ by the equation:

$$\langle u, v\rangle_p^\alpha := \mathcal{I}(\phi_{u,v})$$

The fact that $\mathcal{I}$ is monotonic forces this to once again define an inner product on each tangent space. The fact that $\phi_{u,v}(h)$ is a smooth function of u, v and h, shows that this assignment of inner product is done in a smooth way, and hence defines a Riemannian metric on M. To see that this metric is in fact α-invariant, observe that:

$$\begin{aligned}\langle d\alpha_g u, d\alpha_g v\rangle^{\alpha}_{\alpha_g(p)} &:= \mathcal{I}(\phi_{d\alpha_g u, d\alpha_g v}) \\ &= \mathcal{I}(R_g \phi_{u,v}) \\ &= \mathcal{I}(\phi_{u,v}) \ = \ \langle u, v\rangle^{\alpha}_p.\end{aligned}$$

As an application of this, suppose that there is a point $x \in \mathrm{Fix}_\alpha(G)$. There is an *induced action* $d\alpha : G \times T_xM \to T_xM$ of G on the inner product space $\big(T_xM, \langle\ ,\ \rangle^{\alpha}\big)$, given by $d\alpha(g, u) = d\alpha_g u$.

It turns out that $d\alpha$ is what is known as an 'orthogonal action'. Roughly speaking, this means that if a basis $\mathcal{B}$ is chosen for T_xM which is orthonormal with respect to $\langle\ ,\ \rangle^{\alpha}$, then the matrices corresponding to the α_g in this basis will be orthogonal matrices, *i.e.* $[d\alpha_g]^t_{\mathcal{B}} = [d\alpha_g]^{-1}_{\mathcal{B}}$ for all $g \in G$. For our purposes, $M = S^2$, so these tangent spaces are two-dimensional, hence we can identify these induced actions $d\alpha$ of G on T_xM with linear actions of G on $\mathbb{R}^2$.

Another notion we will employ is that of the exponential map. Let $\langle\ ,\ \rangle$ be a Riemannian metric on a smooth manifold M. For every $p \in M$, there is a map $\exp_p : U \to M$ for some open neighborhood U of $0 \in T_pM$. The Riemannian manifold $\big(M, \langle\ ,\ \rangle\big)$ is said to be *complete* (or *geodesically complete*) if each of the maps $\exp_p$ are defined for all $u \in T_pM$. Since this paper is concerned exclusively about S^2 which is complete, we will only consider complete manifolds here.

The map $\exp_p$ has several important properties. Firstly, $\exp_p$ depends on the Riemannian metric. When we have an action α and an α-invariant metric $\langle\ ,\ \rangle^{\alpha}$, we will indicate the dependence of the exponential map on this metric by writing $\exp^{\alpha}_p$. For any fixed point $x \in M$, $\exp^{\alpha}_x : (T_xM, d\alpha) \to (M, \alpha)$ is G-equivariant, *i.e.* $\exp^{\alpha}_x(d\alpha_g u) = \alpha_g \exp^{\alpha}_x(u)$.

The exponential map $\exp_p$ is a local diffeomorphism, but not a diffeomorphism for any compact M because it will not be injective. Let $B_\varepsilon(0) := \Big\{u \in T_pM \mid \sqrt{\langle u, u\rangle_p} < \varepsilon\Big\}$ be the open ball of radius ε centered at $0 \in T_pM$, and define the *injectivity radius* (at p) to be the number

$$R_p := \sup \left\{ \varepsilon > 0 \mid \ \exp_p \big|_{B_\varepsilon(0)} \text{ is injective} \right\}.$$

This value is important for making arguments about the topology of M, and we will use it to prove Lemma 10.

This concludes the preliminaries section.

3. Classification of Compact Subgroups of O_3 up to Conjugacy

We give here a list of representatives for the elements of $C_c(O_3)$. The conclusion of this section is the following:

Theorem 9. *The second column of Table 1 constitutes a system of distinct representatives for $C_c(O_3)$.*

In other words, every compact Lie subgroup $H \leq O_3$ belongs to the conjugacy class of exactly one of the groups in Table 1.

Important Note: *In the context of Theorem 2, the reader should be thinking of $H = F_f(G)$ from Section 2.2.*

Ref. No.	Connected components	Preferred representative	Image under p	Intersection with SO_3	Orientation preserving?	Isomorphism classes of stabilizers
(1)	1	SO_3	SO_3	SO_3	Y	S^1
(2)	2	O_3	SO_3	SO_3	N	O_2
(3)	1	S^1	S^1	S^1	Y	$1, S^1$
(4)	2	$S^1 \times \mathbb{Z}_2$	S^1	S^1	N	$1, \mathbb{Z}_2, S^1$
(5)	2	$O_2 \cong S^1 \rtimes \mathbb{Z}_2$	O_2	O_2	Y	$1, \mathbb{Z}_2, S^1$
(6)	2	$\widehat{O_2} \cong S^1 \rtimes \mathbb{Z}_2$	O_2	S^1	N	$1, \mathbb{Z}_2, O_2$
(7)	4	$O_2 \times \mathbb{Z}_2$	O_2	O_2	N	$1, \mathbb{Z}_2, O_2$
(8)	k	$\mathbb{Z}_k$	$\mathbb{Z}_k$	$\mathbb{Z}_k$	Y	$1, \mathbb{Z}_k$
(9)	$4j$	$\widetilde{\mathbb{Z}_{2(2j)}}$	$\mathbb{Z}_{4j}$	$\mathbb{Z}_{2j}$	N	$1, \mathbb{Z}_{2j}$
(10)	$4j+2$	$\widetilde{\mathbb{Z}_{2(2j+1)}}$	$\mathbb{Z}_{4j+2}$	$\mathbb{Z}_{2j+1}$	N	$1, \mathbb{Z}_2, \mathbb{Z}_{2j+1}$
(11)	$4n$	$\mathbb{Z}_{2n} \times \mathbb{Z}_2$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	N	$1, \mathbb{Z}_2, \mathbb{Z}_{2n}$
(12)	$4n+2$	$\mathbb{Z}_{2n+1} \times \mathbb{Z}_2$	$\mathbb{Z}_{2n+1}$	$\mathbb{Z}_{2n+1}$	N	$1, \mathbb{Z}_{2n+1}$
(13)	$2k$	D_k	D_k	D_k	Y	$1, \mathbb{Z}_2, \mathbb{Z}_k$
(14)	$2k$	$\widehat{D_k}$	D_k	$\mathbb{Z}_k$	N	$1, \mathbb{Z}_2, D_k$
(15)	$8j$	$\widetilde{D_{2(2j)}}$	D_{4j}	D_{2j}	N	$1, \mathbb{Z}_2, D_{2j}$
(16)	$8j+4$	$\widetilde{D_{2(2j+1)}}$	D_{4j+2}	D_{2j+1}	N	$1, \mathbb{Z}_2, (\mathbb{Z}_2)^2, D_{2j+1}$
(17)	$8n$	$D_{2n} \times \mathbb{Z}_2$	D_{2n}	D_{2n}	N	$1, \mathbb{Z}_2, (\mathbb{Z}_2)^2, D_{2n}$
(18)	$8n+4$	$D_{2n+1} \times \mathbb{Z}_2$	D_{2n+1}	D_{2n+1}	N	$1, \mathbb{Z}_2, D_{2n+1}$
(19)	12	$\mathrm{Tet} \cong A_4$	Tet	Tet	Y	$1, \mathbb{Z}_2, \mathbb{Z}_3$
(20)	24	$\mathrm{Tet} \times \mathbb{Z}_2$	Tet	Tet	N	$1, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_3$
(21)	24	$\mathrm{Oct} \cong S_4$	Oct	Oct	Y	$1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$
(22)	24	$\mathrm{Tet}_F \cong S_4$	Oct	Tet	N	$1, \mathbb{Z}_2, (\mathbb{Z}_2)^2, D_3$
(23)	48	$\mathrm{Oct} \times \mathbb{Z}_2$	Oct	Oct	N	$1, \mathbb{Z}_2, (\mathbb{Z}_2)^2, D_3, D_4$
(24)	60	$\mathrm{Ico} \cong A_5$	Ico	Ico	Y	$1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$
(25)	120	$\mathrm{Ico} \times \mathbb{Z}_2$	Ico	Ico	N	$1, \mathbb{Z}_2, (\mathbb{Z}_2)^2, D_3, D_5$

Table 1

After clarifying the notation of Table 1, we will prove Theorem 9 for the cases where H is finite, and then finally for H a compact Lie subgroup of $\dim(H) \geq 1$.

3.1. **Description and Notation of the Preferred Representatives.** This section is for reference, and is *long*. Those readers primarily interested in theory are encouraged to skip ahead to the proof.

(1)/(2) When $[H] = [SO_3]$ or $[O_3]$, these classes each contain only one element, so the preferred representative is SO_3 or O_3 itself.

(3) When $[H] = [S^1]$, the preferred representative is the subgroup of rotations about the z-axis.

(4) When $[H] = [S^1 \times \mathbb{Z}_2]$, the preferred representative is the subgroup generated by rotations about the z-axis, together with the matrix $-I$.

(5) When $[H] = [O_2]$, the preferred representative is the subgroup generated by rotations about the z-axis, together with a rotation of π radians about the x-axis.

(6) When $[H] = [\widehat{O_2}]$, the preferred representative is the subgroup generated by rotations about the z-axis, together with reflection across the yz-plane.

(7) When $[H] = [O_2 \times \mathbb{Z}_2]$, the preferred representative is the subgroup generated by rotations about the z-axis, together with a rotation of π radians about the x-axis, and with $-I$.

For the finite groups, we describe generators according to the following presentation scheme:

(I)	Cyclic	$\langle a \mid a^k = 1 \rangle$
(II)	Dihedral	$\langle a, b \mid a^k = b^2 = (ab)^2 = 1 \rangle$
(III)	Tetrahedral	$\langle a, b \mid a^3 = b^3 = (ab)^2 = 1 \rangle$
(IV)	Octahedral	$\langle a, b \mid a^4 = b^3 = (ab)^2 = 1 \rangle$
(V)	Icosahedral	$\langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$.

For the cyclic and dihedral[5] cases $\mathbb{Z}_k$ and D_k, the generator a will correspond to rotation through $\frac{2\pi}{k}$ radians about the z-axis, and the generator b will correspond to a rotation of π radians about the x-axis. The alternate embeddings $\widetilde{\mathbb{Z}_{2n}}$, $\widetilde{D_{2n}}$ and $\widehat{D_k}$ as well as the extensions $\mathbb{Z}_k \times \mathbb{Z}_2$ and $D_k \times \mathbb{Z}_2$ will reference these generators.

Note: The embellishment $\widetilde{\ }$ indicates the presence of the generator $-a$. The marking $\widehat{\ }$ indicated the presence of $-b$ (this is in keeping with its use in $\widehat{O_2}$ above).

Note: As a general rule k will either be $2n$ or $2n+1$, and if $k = 2n$, then n will be either $2j$ or $2j+1$.

[5]Here the notation D_k for the dihedral groups corresponds to a group of order $2k$, whose underlying cyclic group is $\mathbb{Z}_k$.

(8) When $[H] = [\mathbb{Z}_k]$, the preferred representative is the subgroup generated by the matrix a.

(9)/(10) When $[H] = [\widetilde{\mathbb{Z}_{2n}}]$, the preferred representative is the subgroup generated by the matrix $-a$.

(11)/(12) When $[H] = [\mathbb{Z}_k \times \mathbb{Z}_2]$, the preferred representative is the subgroup generated by the matrices a and $-I$. Note that when k is odd, this group is cyclic and generated by $-a$.

(13) When $[H] = [D_k]$, the preferred representative is the subgroup generated by the matrices a and b.

(14) When $[H] = [\widehat{D_k}]$, the preferred representative is the subgroup generated by the matrices a and $-b$.

(15)/(16) When $[H] = [\widetilde{D_{2n}}]$ ($n = 2j$ or $2j+1$), the preferred representative is the subgroup generated by the matrices $-a$ and b.

(15)/(16)⋆ $[H] = [\widehat{\widetilde{D_{2n}}}]$, This is a special case that will occur in the proof of Theorem 9, and will turn out to be redundant: In this case, the preferred representative is the subgroup generated by the matrices $-a$ and $-b$.

(17)/(18) When $[H] = [D_k \times \mathbb{Z}_2]$, the preferred representative is the subgroup generated by the matrices a, b and $-I$.

(19) When $[H] = [\mathrm{Tet}]$ (The rotational symmetry group of the tetrahedron), the preferred representative is the subgroup generated by the matrices:

$$a = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \& \quad b = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(20) When $[H] = [\mathrm{Tet} \times \mathbb{Z}_2]$, the preferred representative is the subgroup generated by $-I$ and the generators of Tet as above.

(21) When $[H] = [\text{Oct}]$ (the rotational group of the octahedron), the preferred representative is the subgroup generated by the matrices:

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \& \quad b = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(22) When $[H] = [\text{Tet}_F]$, the preferred representative is the subgroup generated by the matrices $-a$ and b, where a, b correspond to the generators of Oct listed above.

For the construction of the equivariant CW decomposition for this group, it will be in our best interest to realize this as the full symmetry group of the tetrahedron. define $A = b^2a^2$ and $B = b$, and $C = -a^2b^2a$. Direct calculation shows that A and B are precisely the generators for Tet, and C provides the extra reflection across an edge.

(23) When $[H] = [\text{Oct} \times \mathbb{Z}_2]$ (the full symmetry group of the octahedron), the preferred representative is the subgroup generated by the matrices a and b of Oct from above, together with $-I$.

(24) When $[H] = [\text{Ico}]$ (the rotation group of the icosahedron), the preferred representative is the subgroup generated by:

$$s = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \& \quad b = \frac{1}{2} \begin{bmatrix} -\varphi^{-1} & -\varphi & 1 \\ \varphi & -1 & -\varphi^{-1} \\ 1 & \varphi^{-1} & \varphi \end{bmatrix}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. If we set $a = sb$, then $\{a, b\}$ can also serve as a generating set for Ico, and this is in keeping with our presentation conventions from Page 20.

(25) When $[H] = [\text{Ico} \times \mathbb{Z}_2]$ (the full symmetry group of the icosahedron), the preferred representative is the subgroup generated by the matrices a and b of Ico above, together with $-I$.

3.2. **Classification of Finite Subgroups up to Conjugacy.** Below we flesh out the details of Theorem 9 for finite $H \leq O_3$. We rely primarily on the four facts from Section 2.4.

Let $|H| < \infty$, and let $p : O_3 \to SO_3$ denote projection of Section 2.4. The image $p(H) \leq SO_3$ is finite, so it must be in exactly one of the 5 conjugacy classes listed in Corollary 7. The 5 conjugacy classes correspond to finite subgroups which have the presentations listed on page 20.

Since these presentations are valid for every group in a given $[p(H)]$ (the actual a and b vary from class to class), we can endow $p(H)$ with such a presentation[6]. The group H surjects onto $p(H)$, and as such must contain elements that map onto the generators a and b of $p(H)$. We argue by cases:

Case I: $p(H) = \langle a \ \mid \ a^k = 1 \rangle$

We show that $[H]$ must be exactly one of reference numbers (8)-(12). Either $-I \in H$ or $-I \notin H$. If $-I \in H$, then Fact 4 applies and $H \cong \mathbb{Z}_k \times \mathbb{Z}_2$ (reference numbers (11) and (12)). Note that this includes the possibility of the trivial group $1 = \mathbb{Z}_1$, where the preferred representative $1 \times \mathbb{Z}_2$ acts by central involution.

Assume now that $-I \notin H$. Since H surjects onto $p(H)$, it must contain a or $-a$. If $a \in H$, then $-a^i \notin H$ for any i for otherwise this would imply $-I \in \mathbb{Z}$. Thus $a \in H$ and $-I \notin H$ implies that $H \leq SO_3$, so $H = p(H) = \mathbb{Z}_k$ (class number (8)).

If $-a \in H$ and $-I \notin H$, then k must be even. To see this, notice that if $k = 2n+1$, then $(-a)^k = -a^k = -I$. Assume then that $k = 2n$. Since $-a$ satisfies the same relations as a, *i.e.* $(-a)^k = 1$, it must generate a cyclic group of order k. The representative for this class is $\widetilde{\mathbb{Z}_{2n}}$ (reference numbers (9) and (10)).

To see that these conjugacy classes are all distinct, notice that $\mathbb{Z}_{2n+1}$ has odd order, and so cannot be conjugate to any of the other possibilities found here in case (I). By Fact 3, $\mathbb{Z}_{2n}$ cannot be conjugate to $\widetilde{\mathbb{Z}_{2n}} \nsubseteq SO_3$ nor $\mathbb{Z}_n \times \mathbb{Z}_2 \nsubseteq SO_3$, because $\mathbb{Z}_{2n} \leq SO_3$. Finally $[p(\widetilde{\mathbb{Z}_{2n}})] = [\mathbb{Z}_{2n}] \neq [\mathbb{Z}_n] = [p(\mathbb{Z}_n \times \mathbb{Z}_2)]$, so by Fact 1 $[\widetilde{\mathbb{Z}_{2n}}] \neq [\mathbb{Z}_n \times \mathbb{Z}_2]$.

Case II: $p(H) = \langle a, b \ \mid \ a^k = b^2 = (ab)^2 = 1 \rangle$

We show that $[H]$ must be exactly one of reference numbers (13)-(18). H may or may not contain $-I$, must contain a or $-a$ or both, and must contain b or $-b$ or both. If $-I \in H$,

[6]There is no harm in reading this section as though $p(H)$ is in fact the preferred representative of $[p(H)]$ and it is recommended that the reader consider this situation if something becomes difficult to visualize.

then $a, b, -I \in H$ and $|H| = 4k$. By Fact 4, the preferred representative in this case is $D_k \times \mathbb{Z}_2$ for $k = 2n$ or $= 2n+1$ (reference numbers (17) and (18)). As with the cyclic case, if $k = 2n+1$, then $-a \in H$ implies $-I \in H$, so $a, b, -I \in H$, and no new cases are obtained.

If $-I \notin H$, and $-a \in H$ then k must be even, say $k = 2n$. From here there are two possibilities: $-b \in H$ or $-b \notin H$. If $-b \notin H$ then $H = \langle(-a), b\rangle$, and these generators satisfy all the same relations as a and b, so H is dihedral. The preferred representative for this class is $\widetilde{D_{2n}}$ (reference numbers (15) and (16)). If $-b \in H$ then $H = \langle(-a), (-b)\rangle$, and these generators satisfy all the same relations as a and b, so H is dihedral. The preferred representative for this class is $\widehat{\widetilde{D_{2n}}}$ (this will turn out to be the same as reference numbers (15) and (16)).

If $-I, -a \notin H$, but $-b \in H$, then $H = \langle a, (-b)\rangle$ and H is dihedral as we have seen before. The preferred representative in this case is $\widehat{D_n}$ (class number (14)). Finally, if none of $-I, -a, -b$ are in H, then $H = p(H)$ (class number (13)). This exhausts all the possibilities when $p(H) \cong D_k$

Now we must show that these conjugacy classes are all distinct. The only classes of singly-even $(2(2n+1))$ order groups are $[D_{2n+1}]$ (number (13) for n odd) and $[\widehat{D_{2n+1}}]$ (number (14) for n odd). $D_{2n+1} \leq SO_3$ and $\widehat{D_{2n+1}} \nleq SO_3$, so these must represent distinct conjugacy classes by Fact 3.

Of the classes whose groups are doubly-even ($\equiv 0 \mod 4$), $D_{2n+1} \times \mathbb{Z}_2$ is the only group that projects onto a group of singly-even order: $p(D_{2n+1} \times \mathbb{Z}_2) = D_{2n+1}$, so it (class number (18)) must be distinct from the rest.

Of the classes that remain, $[D_{2n}]$ (number (13), n even) is the only class whose groups are contained in SO_3, so it must be distinct from the rest.

Of the classes that remain, $[D_{2n} \times \mathbb{Z}_2]$ is the only class whose groups contain $-I$, so by Fact 2 it (class number (17)) must be distinct from the rest.

The only remaining classes are $[\widetilde{D_{2n}}]$, $[\widehat{\widetilde{D_{2n}}}]$ and $[\widehat{D_{2n}}]$. We will show that $[\widehat{D_{2n}}]$ is a distinct class, but $[\widehat{\widetilde{D_{2n}}}] = [\widetilde{D_{2n}}]$.

Denote the intersection of SO_3 with the preferred representatives of these last three classes by $\widehat{H_0}, \widehat{\widehat{H_0}}$ and $\widetilde{H_0}$ respectively. The generators for these groups are

$$\widehat{H_0} = \langle a \rangle \cong \mathbb{Z}_{2n}$$
$$\widehat{\widehat{H_0}} = \langle a^2, ab \rangle \cong D_n$$
$$\widetilde{H_0} = \langle a^2, b \rangle \cong D_n.$$

For $\widehat{D_{2n}}$, the generator $-b$ has been left out by intersecting with SO_3, and this makes $\widehat{H_0}$ cyclic, so by Fact 3 it (class number (14), even) must be in a distinct conjugacy class from the remaining two.

To see that $[\widehat{\widehat{D_{2n}}}] = [\widetilde{D_{2n}}]$, create a square root $a^{1/2}$ of the generator[7] a as in Section 2.4. First observe the geometric fact that $a^{-1/2}(b)a^{-1/2} = b$. We remark that this cannot be proven algebraically in terms of generators, and relies entirely on the construction of $a^{1/2}$ *as an element of* SO_3. Next observe that $a^{1/2}(-a)a^{-1/2} = -a$, and

$$a^{1/2}(b)a^{-1/2} = aa^{-1/2}\, b\, a^{-1/2} = ab.$$

Now we calculate:

$$\begin{aligned}
[\widetilde{D_{2n}}] &= [\langle -a, b \rangle] \\
&= [a^{1/2} \cdot \langle -a, b \rangle \cdot a^{-1/2}] \\
&= [\langle a^{1/2}(-a)a^{-1/2} \ , \ a^{1/2}(b)a^{-1/2} \rangle] \\
&= [\langle -a, ab \rangle] \\
&= [\langle -a \ , \ (-a)^{-1} \cdot ab \rangle] \\
&= [\langle -a \ , \ -a^{-1} \cdot ab \rangle] \\
&= [\langle -a, -b \rangle] \\
&= [\widehat{\widehat{D_{2n}}}]
\end{aligned}$$

[7]The generator a is the same for both preferred representatives of these two classes.

Thus we see that both $\widehat{\widetilde{D_{2n}}}$ and $\widetilde{D_{2n}}$ belong to class number (15) when n is even, or (16) when n is odd, and this concludes the section on the dihedral classes of $C_c(O_3)$.

Case III: $p(H) = \langle a, b \mid a^3 = b^3 = (ab)^2 = 1\rangle$

We show that $[H]$ must be either class number (19) or (20). In this case, H must contain a or $-a$ or both, and b or $-b$ or both. If H contains either $-a$ or $-b$, then it must contain $-I$, and hence $H = \langle a, b, -I\rangle \cong \text{Tet} \times \mathbb{Z}_2$ (class number (20)). If H contains neither $-a$, nor $-b$, then $-I \notin H$ and $H \cong$ Tet itself (class number (19)).

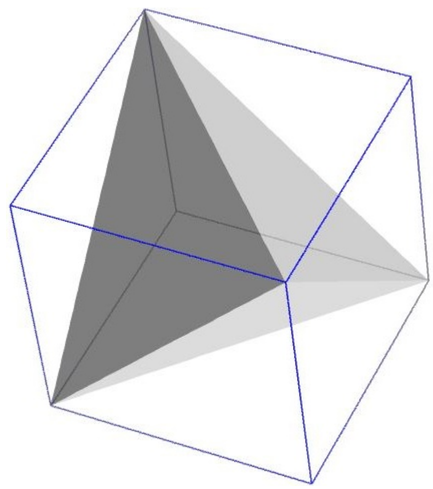
The groups have different orders, so they must represent distinct conjugacy classes.

Case IV: $p(H) = \langle a, b \mid a^4 = b^3 = (ab)^2 = 1\rangle$

We show that $[H]$ must be class number (21), (22) or (23). If H contains $-b$, then $-I \in H$ and $H \cong \text{Oct} \times \mathbb{Z}_2$ (class number (23)). If H contains neither $-a$, nor $-b$, then $-I \notin H$ and $H \cong$ Oct itself (class number (21)). If H doesn't contain $-b$, but does contain $-a$, then $(-a)$ and b satisfy the same relations as a and b, and hence generate a subgroup isomorphic to $\text{Oct} \cong S_4$ (class number (22)). The preferred representative for this last group is Tet_F.

The group $\text{Oct} \times \mathbb{Z}_2$ clearly lies in a distinct class from the other two groups, because its order is twice that of the other two. The remaining classes must be distinct by Fact 3, because $\text{Oct} \leq SO_3$, and $\text{Tet}_F \nleq SO_3$.

We remark that this alternate, orientation reversing embedding of Oct is referred to as Tet_F because it represents the full symmetry group of a tetrahedron. To visualize this, consider the tetrahedron inscribed inside the cube below, and remember that the cube and the octahedron have the same symmetry group by duality.



Case V: $p(h) = \langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$

If H contains either $-a$ or $-b$, then $-I \in H$ and $H \cong \text{Ico} \times \mathbb{Z}_2$ (class number (25)). Otherwise $-I \notin H$ and $H \cong \text{Ico}$ (class number (24)).

The classes are necessarily distinct, because their representatives have different orders.

This concludes the proof of Theorem 9 for the case of finite H.

□

3.3. **Classification of Compact Lie Subgroups of Positive Dimension up to Conjugacy.** Let $H \leq O_3$ be a compact Lie subgroup of $\dim(H) \geq 1$. By Proposition 8, $p(H)$ is conjugate to one of S^1, O_2 or SO_3. We handle each of these cases separately:

Case I: $[p(H)] = [SO_3]$

In this case H is clearly either O_3 or SO_3, and these types have unique, distinct conjugacy classes (1) and (2).

Case II: $[p(H)] = [S^1]$

We show that $[H]$ must be either class number (3) or (4). Let H_0 be the connected component of the identity in H. H_0 is a compact, connected one-dimensional manifold, and as such diffeomorphic to S^1. Since SO_3 is connected, $H_0 \leq SO_3$. This implies that

$p(H_0) = H_0$, and we find:

$$S^1 \cong p(H) \supseteq p(H_0) = H_0 \cong S^1$$

$$\Longrightarrow$$

$$p(H) = H_0 \leq H$$

If $H \leq SO_3$, then all three of H, H_0 and $p(H)$ groups are the same copy of S^1, so H is conjugate the preferred representative of S^1 (class number (3)).

If $H \nsubseteq SO_3$, then there is some $g \in H \setminus SO_3$. Then $p(g) \in H$, and

$$g^{-1}p(g) = g^{-1}\big(\det(g)g\big) = -I \in H.$$

By Fact 4, $H \cong S^1 \times \mathbb{Z}_2$. Any group in this class is completely determined by it's axis of rotation, and conjugacy is realized by rotations from one axis to the other.

Case III: $[p(H)] = [O_2]$

We show that $[H]$ must be class number (5), (6) or (7). From our analysis in Section 2.4, $p(H)$ has a subgroup isomorphic to S^1, and an element b of order two that swaps the poles where the axis of rotation intersects S^2. As we have seen in the previous case (II: $[p(h)] = [S^1]$), the connected component of the identity H_0 is diffeomorphic to S^1. We distinguish three cases.

Either $-I \in H$ or not. If $-I \in H$, then Fact 4 applies and $H \cong O_2 \times \mathbb{Z}_2$ (class number (7)).

Suppose then that $-I \notin H$. H must contain either b or $-b$ but cannot contain both, because $(-b)b = -I$. If $b \in H$, then $H \subseteq SO_3$, so $[H] = [O_2]$ (class number (5)). If $-b \in H$, then every element of H preserves the poles of rotation, and the preferred representative in this case is $\widehat{O_2}$ (class number (6)).

The group $O_2 \times \mathbb{Z}_2$ has four connected components, and as such cannot be conjugate to the other two. Using Fact 3, $O_2 \leq SO_3$ and $\widetilde{O_2} \nleq SO_3$ cannot be conjugate to one another.

This concludes the proof of Theorem 9 for the case of $\dim(H) \geq 1$.

■

4. Construction of Equivariant Skeleta for the Finite Subgroups

Below we list the equivariant decomposition of S^2 corresponding to the preferred representatives of each conjugacy class in $C_c(O_3)$. The decompositions corresponding to the cyclic and dihedral cases require repeated reference to specific subsets of S^2, so we begin by outlining some notational conventions.

4.1. **Preliminaries for Cyclic and Dihedral Cases.**

Here we make heavy use of the explicit generators a and b of the preferred representatives of the cyclic and dihedral classes (8)-(18) as described in Section 3.1. The North pole N and the South pole S will always be in the 0-skeleton. They will be in the same orbit whenever H contains $-1, b$ or $-a$. The point p on the equator will usually be $(1, 0, 0)$, but for all groups with a $\widehat{\ }$ decoration p will be $(0, 1, 0)$. This alternate choice is necessary for the reflective dihedral cases only, and we will indicate when this is the case.

As a shorthand for referring to certain cells that appear frequently, define the following:

- M_r will be the meridional arc connecting N to S, passing through $a^r \cdot p$, and $M := M_0$.
- M_r^+ will be the meridional arc connecting N to $a^r \cdot p$, and $M^+ := M_0^+$.
- E_r will be the equatorial arc connecting p to $a^r \cdot p$, and $E := E_1$.
- T_r will be the triangular cell spanned by N, p and $a^r \cdot p$, and $T := T_1$ (having positive orientation).
- B_r will be the bigon spanned by M and $a^r \cdot M$, and $B := B_1$ (having positive orientation).

4.2. **Cyclic Cases and Their Extensions.** $H = \mathbb{Z}_n$**:**

For the orientation preserving cyclic cases (even or odd order), we have:

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left({}^{H}\!/_{H}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(S) \cong \left({}^{H}\!/_{H}\right) \times D^0$$

(1) -cell:

$$\mathrm{Orb}_\Lambda(M) \cong \left(H\diagup_1\right) \times D^1$$

(2) -cell:

$$\mathrm{Orb}_\Lambda(B) \cong \left(H\diagup_1\right) \times D^2$$

$H = \widetilde{\mathbb{Z}_{2n}}$**:**

For this group, there are different decompositions depending on whether n is even or odd. This phenomenon comes from the fact that a^n is rotation by π radians, so $-a^n$ corresponds to reflection across the xy-plane. When n is odd, $-a^n \in \widetilde{\mathbb{Z}_{2n}}$, but when n is even, $-a^n \notin \widetilde{\mathbb{Z}_{2n}}$.

For n even, we have

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left(H\diagup_{\langle a^2\rangle}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(p) \cong \left(H\diagup_1\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(M^+) \cong \left(H\diagup_1\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(E) \cong \left(H\diagup_1\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(T_2) \cong \left(H\diagup_1\right) \times D^2$$

For n odd, we have

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left(H\diagup_{\langle a^2\rangle}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(p) \cong \left(H\diagup_1\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(M^+) \cong \left({}^{H}\!\diagup_{1}\right) \times D^1$$

$$\mathrm{Orb}_\Lambda(E_2) \cong \left({}^{H}\!\diagup_{\langle -a^n \rangle}\right) \times D^1$$

(2) -cell:

$$\mathrm{Orb}_\Lambda(T_2) \cong \left({}^{H}\!\diagup_{1}\right) \times D^1$$

$H = \mathbb{Z}_{2n} \times \mathbb{Z}_2$:

For this group, we have:

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left({}^{H}\!\diagup_{\langle a \rangle}\right) \times D^0$$

$$\mathrm{Orb}_\Lambda(p) \cong \left({}^{H}\!\diagup_{\langle -a^n \rangle}\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(M^+) \cong \left({}^{H}\!\diagup_{1}\right) \times D^1$$

$$\mathrm{Orb}_\Lambda(E) \cong \left({}^{H}\!\diagup_{\langle -a^n \rangle}\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(T) \cong \left({}^{H}\!\diagup_{\langle 1 \rangle}\right) \times D^2$$

$H = \mathbb{Z}_{2n+1} \times \mathbb{Z}_2$:

For this case, we have:

(0) -cells:

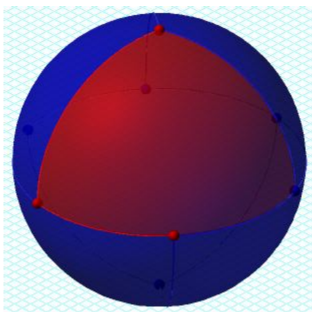
$$\mathrm{Orb}_\Lambda(N) \cong \left({}^{H}\!\diagup_{\langle a \rangle}\right) \times D^0$$

(1) -cells:

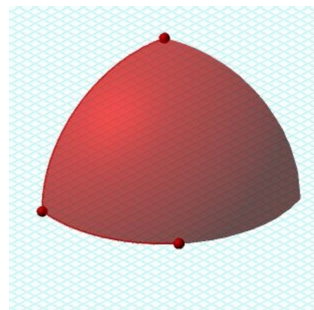
$$\mathrm{Orb}_\Lambda(M) \cong \left({}^{H}\!\diagup_{1}\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(B_{1/2}) \cong \left({}^{H}\!/_{1}\right) \times D^2$$



$\mathbb{Z}_3 \times \mathbb{Z}_2$



Full skeleton

Cells in the orbit space

4.3. **Dihedral Cases and Their Extensions.**

$H = D_n$**:**

For the orientation preserving dihedral cases D_n, we have

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left({}^{H}\!/_{\langle a\rangle}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(p) \cong \left({}^{H}\!/_{\langle b\rangle}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(a^{1/2} \cdot p) \cong \left({}^{H}\!/_{\langle ab\rangle}\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(M^+) \cong \left({}^{H}\!/_{1}\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(E_{1/2}) \cong \left({}^{H}\!/_{1}\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(T) \cong \left({}^{H}\!/_{1}\right) \times D^2$$

$H = \widehat{D_n}$**:**

For these groups, choose $p = (0,1,0)$. Then we have:

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left({}^{H}\!\diagup_{H}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(S) \cong \left({}^{H}\!\diagup_{H}\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(M) \cong \left({}^{H}\!\diagup_{\langle -b\rangle}\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(M_1) \cong \left({}^{H}\!\diagup_{\langle -ab\rangle}\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(B_{1/2}) \cong \left({}^{H}\!\diagup_{1}\right) \times D^2$$

Just as with the $\mathbb{Z}_{2n}$ case, the orientation reversing dihedral cases corresponding to the generator $-a$, will differ depending on whether n is even or odd.

$H = \widetilde{D_{2n}}$ **(**n **even):**

For n even, we have:

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left({}^{H}\!\diagup_{\langle a^2, -ab\rangle}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(p) \cong \left({}^{H}\!\diagup_{\langle b\rangle}\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(M^+) \cong \left({}^{H}\!\diagup_{1}\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(M_{1/2}) \cong \left({}^{H}\!\diagup_{\langle -a^{n+1}b\rangle}\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(B_{1/2}) \cong \left({}^{H}\!\diagup_{1}\right) \times D^2$$

$H = \widetilde{D_{2n}}$ **(n odd):**

For n odd, we have: we have:

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left(H/\langle a^2, -ab\rangle\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(p) \cong \left(H/\langle -a^n, b\rangle\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(a \cdot p) \cong \left(H/\langle -a^n, a^2 b\rangle\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(M^+) \cong \left(H/\langle -a^n b\rangle\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(M_1^+) \cong \left(H/\langle -a^{n+2} b\rangle\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(E) \cong \left(H/\langle -a^n\rangle\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(T) \cong \left(H/1\right) \times D^2$$

$H = D_{2n} \times \mathbb{Z}_2$**:**

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left(H/\langle a, -b\rangle\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(p) \cong \left(H/\langle -a^n, b\rangle\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(a^{1/2} \cdot p) \cong \left(H/\langle -a^n, ab\rangle\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(M^+) \cong \left(H/\langle -a^n b\rangle\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(M_{1/2}^+) \cong \left(H/\langle -a^{n+1} b\rangle\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(E_{1/2}) \cong \left(H/\langle -a^n\rangle\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(T_{1/2}) \cong \left({}^{H}\!\diagup_{1}\right) \times D^2$$

$H = D_{2n+1} \times \mathbb{Z}_2$**:**

(0) -cells:

$$\mathrm{Orb}_\Lambda(N) \cong \left({}^{H}\!\diagup_{\langle a, -b\rangle}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(p) \cong \left({}^{H}\!\diagup_{\langle b\rangle}\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(M^+) \cong \left({}^{H}\!\diagup_{1}\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(M_{1/4}) \cong \left({}^{H}\!\diagup_{\langle -a^{n+1}b\rangle}\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(B_{1/4}) \cong \left({}^{H}\!\diagup_{1}\right) \times D^2$$

4.4. **Polyhedral Cases.** Duality arguments show that the 5 platonic solids give rise to 3 symmetry groups: icosahedral/dodecahedral, cubic/octahedral, and tetrahedral (the tetrahedron is self-dual). By inscribing a platonic solid in S^2, we can use projection from the origin to map the surface of the solid homeomorphically onto S^2. This allows S^2 to naturally inherit the structure of a H-CW complex. If H is one of the three polyhedral groups, then by choosing the icosahedron, octahedron, and tetrahedron, the corresponding CW complex can be assumed to have triangular 2-cells. It should be noted that this is not the cell decomposition we will use, but it is the starting point from which we construct our desired decomposition.

H **is any of the orientation preserving polyhedral groups.** For these cases, we take the first barycentric subdivision of each triangular face. Then combine two adjacent triangles if they come from the same original triangle and both contain a common vertex of the original triangle in their boundaries. This will result in a decomposition of each face into three congruent quadrilateral faces. Using the generators a and b described in Section 3.1, label the vertices v_a, v_b and v_{ab} according to the following condition:

$$\mathrm{Stab}_\Lambda(v_a) = \langle a\rangle, \quad \mathrm{Stab}_\Lambda(v_b) = \langle b\rangle \quad \& \quad \mathrm{Stab}_\Lambda(v_{ab}) = \langle ab\rangle$$

Since a, b and $ab \in SO_3$, they are each a rotation. Thus they each have exactly two fixed points in S^2, and in the surface of the inscribed polyhedron. In order to determine v_a, v_b explicitly, choose them according to the right-hand rule with both a and b corresponding to counter-clockwise rotations about v_a and v_b resp. There is exactly one triangle of the barycentric subdivision which contains v_a and v_b, and has as its other vertex v_{ab}, and this is how v_{ab} is determined.

There are two triangles in the barycentric subdivision that share both v_a and v_b as common vertices. These two triangles join together to form a quadrilateral which we denote by Q_0. Let $J_{a,0}$ be the line segment connecting v_{ab} to v_b, and $J_{b,0}$ be the line segment connecting v_a to v_{ab}. Finally let Q, J_a, J_b, u_a, u_{ab} and u_b be the projections of Q_0, $J_{a,0}$, $J_{b,0}$, v_a, v_{ab} and v_b respectively onto the sphere.

The H-CW decompositions are as follows:

$H =$ **Tet or Oct or Ico:**

(0) -cells:

$$\mathrm{Orb}_\Lambda(u_a) \cong \left(H/\langle a\rangle\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(u_{ab}) \cong \left(H/\langle ab\rangle\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(u_b) \cong \left(H/\langle b\rangle\right) \times D^0$$

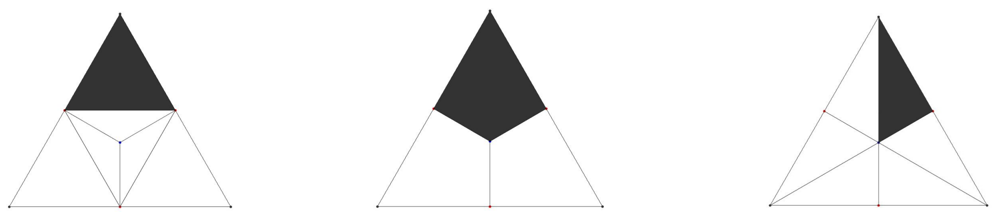
(1) -cells:

$$\mathrm{Orb}_\Lambda(J_a) \cong \left(H/1\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(J_b) \cong \left(H/1\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(Q) \cong \left(H/1\right) \times D^2$$

Figure 2



Tet $\times \mathbb{Z}_2$ Orientation preserving Orientation reversing

H is any of the orientation reversing polyhedral groups except for Tet $\times \mathbb{Z}_2$: Here we take the first barycentric subdivision of each triangular face. Then determine the points v_a, v_{ab} and v_b as before, but do not combine the triangles. Define $J_{a,0}$, $J_{b,0}$ as before, and also define $J_{ab,0}$ to be the line segment connecting v_a to v_b. Now project v_a, v_{ab}, v_b, $J_{a,0}$, $J_{ab,0}$ and $J_{b,0}$ onto the sphere to form u_a, u_{ab}, u_b, J_a, J_{ab} and J_b respectively. The three arcs J_i bound a spherical triangle that will be denoted Δ, and this will be our fundamental domain for the action.

$H =$ **Oct** $\times \mathbb{Z}_2$**:**

(0) -cells:

$$\mathrm{Orb}_\Lambda(u_a) \cong \left({H}\big/{\langle a, -ba^2b^2a\rangle}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(u_{ab}) \cong \left({H}\big/{\langle ab, -a^2b^2a\rangle}\right) \times D^0$$
$$\mathrm{Orb}_\Lambda(u_b) \cong \left({H}\big/{\langle b, -ba^2b^2a\rangle}\right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(J_a) \cong \left({H}\big/{\langle -a^2b^2a\rangle}\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(J_{ab}) \cong \left({H}\big/{\langle -ba^2b^2a\rangle}\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(J_b) \cong \left({H}\big/{\langle -ba^2b^2\rangle}\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(\Delta) \cong \left(H \big/ 1 \right) \times D^2$$

For the case of the full symmetry group of the tetrahedron, we apply the above construction to the faces of the tetrahedron, and use the generators A and B to determine the cells u_A, u_{AB}, u_B, J_A, J_{AB}, J_B and Δ. Together, A and B generate Tet, so we must also include the third generator C from Section 3.1. Note that $\{I, AC, CB, AB\}$ is isomorphic to the Klein four group. The resultant equivariant cells are then:

$H = \mathbf{Tet}_F$**:**

(0) -cells:

$$\mathrm{Orb}_\Lambda(u_A) \cong \left(H \big/ \langle A, C \rangle \right) \times D^0$$
$$\mathrm{Orb}_\Lambda(u_{AB}) \cong \left(H \big/ \langle AC, CB \rangle \right) \times D^0$$
$$\mathrm{Orb}_\Lambda(u_B) \cong \left(H \big/ \langle B, C \rangle \right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(J_A) \cong \left(H \big/ \langle AC \rangle \right) \times D^1$$
$$\mathrm{Orb}_\Lambda(J_{AB}) \cong \left(H \big/ \langle C \rangle \right) \times D^1$$
$$\mathrm{Orb}_\Lambda(J_B) \cong \left(H \big/ \langle CB \rangle \right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(\Delta) \cong \left(H \big/ 1 \right) \times D^2$$

$H = \mathbf{Ico} \times \mathbb{Z}_2$**:**

(0) -cells:

$$\mathrm{Orb}_\Lambda(u_a) \cong \left(H \big/ \langle a, -b^2ab^2a^2b^2a^2 \rangle \right) \times D^0$$
$$\mathrm{Orb}_\Lambda(u_{ab}) \cong \left(H \big/ \langle ab, -bab^2a^2b^2a^2 \rangle \right) \times D^0$$
$$\mathrm{Orb}_\Lambda(u_b) \cong \left(H \big/ \langle b, -b^2ab^2a^2b^2a^2 \rangle \right) \times D^0$$

(1) -cells[8]:

$$\mathrm{Orb}_\Lambda(J_a) \cong \left(H/\langle -bab^2a^2b^2a^2\rangle\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(J_{ab}) \cong \left(H/\langle -b^2ab^2a^2b^2a^2\rangle\right) \times D^1$$
$$\mathrm{Orb}_\Lambda(J_b) \cong \left(H/\langle -ab^2ab^2a^2b^2a^2\rangle\right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(\Delta) \cong \left(H/1\right) \times D^2$$

$H =$ **Tet** $\times \mathbb{Z}_2$**:** [9]

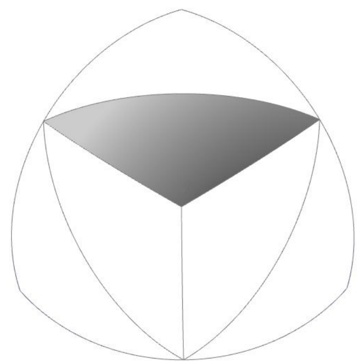
For this case, we begin by dividing up each 2-simplex into 6 smaller 2-simplices. Start by connecting the barycenter of each triangular face to the centers of each of its edges using three straight line segments. Then connect the centers of each of the original edges to one another using straight line segments. The reader may notice at this point that the triangles in this decomposition are not all of the same size, and so cannot be contained in a single orbit under any $H \leq O_3$. Fortunately when the tetrahedron is inscribed inside S^2, each of these triangles has the same solid angle, so upon projection onto the sphere, these result in spherical triangles that all have the same area.

Using the generators a and b from Section 3.1, we aim to single out a specific spherical triangle as our fundamental domain. The element $a \in SO_3$ is a rotation of order three, and must have two poles $\{\pm u\} \subseteq S^2$ in it's 1-eigenspace. The element $ab^2a = a^2ba^2 = ba^2b = b^2ab^2 \in \mathrm{Tet} \leq SO_3$ has order two, so it must correspond to a rotation of π radians about some axis ℓ. This shows that the element $-ab^2a \in \mathrm{Tet} \times \mathbb{Z}_2$ corresponds to reflection across the plane $\ell^\perp$. Evidently there are precisely two spherical triangles that have a vertex $u_a \in \{\pm u\}$ whose opposite arc J_a is fixed by the element $-ab^2a$. Choose one of these spherical triangles and label it Δ', then the triangle not chosen will be $-I \cdot \Delta'$.

[8]I am particularly grateful to Daniel Flores for helping me determine these stabilizers.

[9]The reason this case is set apart from the others is essentially coming from the fact that the tetrahedron is self-dual. This causes $-I$ to send centers of faces to vertices and *vice versa*, an effect that does not occur in any of the other cases.

FIGURE 4. A fundamental domain for Tet $\times \mathbb{Z}_2$



Of the other vertices of Δ', one of them is stabilized by both ab and $-ba$, and hence by $K := \langle ab, -ba\rangle$ which is isomorphic to the Klein four group. Label this vertex u_K, and label the arc connecting u_a to u_K as J_K. With all of this notation, the H-CW decomposition is:

$H = \mathbf{Tet} \times \mathbb{Z}_2$**:**

(0) -cells:

$$\mathrm{Orb}_\Lambda(u_a) \cong \left(H / \langle a \rangle \right) \times D^0$$

$$\mathrm{Orb}_\Lambda(u_K) \cong \left(H / \langle ab, -ba \rangle \right) \times D^0$$

(1) -cells:

$$\mathrm{Orb}_\Lambda(J_a) \cong \left(H / \langle -ab^2 a \rangle \right) \times D^1$$

$$\mathrm{Orb}_\Lambda(J_K) \cong \left(H / \langle 1 \rangle \right) \times D^1$$

(2) -cells:

$$\mathrm{Orb}_\Lambda(\Delta') \cong \left(H / 1 \right) \times D^2$$

We comment here that the one-skeleton of quotient space can be taken to be contractible in the cases where the action is orientation preserving. This is useful in classifying G-equivariant vector bundles over the two sphere and in determining whether or not such bundles have algebraic models, for more on this see [16]. An approach for this line of reasoning relies on techniques developed by Hambleton and Hausmann in [6] and [7]. Unfortunately, the one-skeleton of the quotient space will not be contractible in the case of the orientation reversing actions. Although these remarks do not affect the proof of Theorem 2, these types of questions served as a primary motivation to determine the validity of Theorem 2.

5. Proof of Theorem 2

Let $\alpha : G \times S^2 \to S^2$ be a smooth action, and let $f : S^2 \to S^2$ be the conjugating function provided by Theorem 1. Define $H := F_f(G) \leq O_3$ as in Section 2.2. By our observations in Section 2.1, we may assume that H is the preferred representative of $[H] \in C_c(O_3)$, and that the resulting action λ of G on S^2 corresponds (via composition with F_f) to the standard action Λ of H on S^2. We index the cases of the proof by the possible preferred representatives H. Some of these cases have proofs that are identical, and so the representatives are grouped by the properties (such as transitivity of Λ) of H that allow for certain arguments.

5.1. **Proof of Theorem 2 if $H = O_3$ or SO_3.**

In these cases, Λ and hence α and λ are transitive actions, and this implies that f was smooth to begin with. To see this, let $p \in S^2$. If $q \in S^2$, then by transitivity there is some $g \in G$ such that $q = \alpha_g p$. By the equivariance of f, we have that

$$f(q) = f(\alpha_g p) \;=\; \lambda_g f(p).$$

This shows that f is completely determined by the value $f(p)$ and the linear action λ. Let $K := \mathrm{Stab}_\alpha(p)$ be the stabilizer of p under α. Conjugating by f shows that this is equal to the stabilizer of $f(p)$ under λ. By the Smooth Orbit/Stabilizer Theorem, we obtain diffeomorphisms

$$\Phi_\alpha : {}^{G}\!/_{K} \to S^2 \qquad\qquad \Phi_\lambda : {}^{G}\!/_{K} \to S^2$$
$$gK \mapsto \alpha_g p \qquad\qquad gK \mapsto \lambda_g f(p).$$

Thus f can be written as the composition

$$\begin{array}{ccccc} S^2 & \xrightarrow{\Phi_\alpha^{-1}} & {}^{G}\!/\!_{K} & \xrightarrow{\Phi_\lambda} & S^2 \\ \alpha_g p & \longmapsto & gK & \longmapsto & \lambda_g f(p), \end{array}$$

and this establishes the cases where H acts transitively. In terms of conjugacy classes, this proves both the $H \cong O(3)$ and $H \cong SO(3)$ cases.

□

5.2. **Proof of Theorem 2 if $H = S^1$ or $\widehat{O_2}$.**

These cases are grouped together because they have two points, the north $N_0 = (0,0,1)$ and south $S_0 = (0,0,-1)$ poles, that are fixed under the action λ by every element of G. Let $p_0 = (1,0,0)$, and let $\gamma : [0,1] \to S^2$ be a unit speed parameterization of the unique meridian which passes through p_0 and connects the north pole N_0 to the south pole S_0. Define $p := f^{-1}(p_0)$, $N := f^{-1}(N_0)$ and $S := f^{-1}(S_0)$, so that N, S are the unique isolated fixed points of the action α.

Using the exponential map $\exp_N^\alpha$, we can map a closed disk D_N centered at $0 \in T_N S^2$ diffeomorphically onto a closed neighborhood of N.

Lemma 10. *The radius r of $D_N = \overline{B_r(0)}$ can be increased until $p \in \exp_N^\alpha(\partial D_N)$.*

Proof. Let $R := R_N$ be the injectivity radius of $\exp_N^\alpha$. Since $\exp_N^\alpha$ is injective on $B_R(0)$, but not on $\overline{B_R(0)}$, there exist $u, v \in \partial B_R(0)$ with $u \neq v$ and $\exp_N^\alpha(u) = \exp_N^\alpha(v) =: x_0 \in S^2$.

The induced action $d\alpha$ of $S^1 \leq G$ on $T_N S^2$ is orthogonal with respect to the metric $\langle\ ,\ \rangle^\alpha$ by construction, so two vectors in $T_N S^2$ are in the same orbit under $d\alpha$ if and only if they have the same norm with respect to $\langle\ ,\ \rangle^\alpha$. Since $\langle u, u\rangle^\alpha = \langle v, v\rangle^\alpha = R^2$, there must be some $g \in S^1$ such that $d\alpha_g u = v$. Note that $u \neq v$ implies that $g \neq 1 \in S^1$. Using the equivariance of the exponential function, we find that

$$\begin{aligned}\alpha_g x_0 &= \alpha_g \exp_N^\alpha(u) \\ &= \exp_N^\alpha(d\alpha_g u) \\ &= \exp_N^\alpha(v) \;=\; x_0 \\ &\Longrightarrow \\ x_0 \in &\mathrm{Fix}_\alpha(g) \;=\; \{N, S\}\end{aligned}$$

By the continuity of $\exp_N^\alpha$ and the definition of R, x_0 cannot be N, and therefore $x_0 = S$. For any $w \in \partial B_R(0)$, there is a unique $h \in S^1$ such that $d\alpha_h u = w$. This shows that

$$\begin{aligned}\exp_N^\alpha(w) &= \exp_N^\alpha(d\alpha_h u) \\ &= \alpha_h \exp_N^\alpha(u) \\ &= \alpha_h S \;=\; S.\end{aligned}$$

Therefore $\exp_N^\alpha$ maps the entire boundary $\partial B_R(0)$ onto S.

The image of $\exp_N^\alpha$ is the quotient of a disk where the boundary is collapsed to a point, and this is topologically a sphere. Taking this quotient followed by set inclusion, we find that on the closed disk, $\exp_N^\alpha$ can be factored as:

$$\overline{B_R(0)} \twoheadrightarrow \mathrm{im}(\exp_N^\alpha) \;\cong\; \overline{B_R(0)}\big/\partial B_R(0) \;\cong\; S^2 \hookrightarrow S^2.$$

Any injective map from S^2 to itself must also be surjective, so $p \in \exp_N^\alpha(B_R(0))$. Now let $u \in B_R(0)$ be such that $\exp_N^\alpha(u) = p$. If we set $r = \|u\|$, then $p = \exp_N^\alpha(u) \in \exp_N^\alpha(\partial B_r(0)) = \exp_N^\alpha(\partial D_N)$

□

We now continue with the proof of Theorem 2 when $H = S^1$ or $\widehat{O_2}$. Note that $\exp_N^\alpha(\partial D_N) = \partial \exp_N^\alpha(D_N)$ is precisely the orbit of $f(p)$ under α. We perform a similar construction for S, and find that $\exp_S^\alpha(D_S)$ and $\exp_N^\alpha(D_N)$ meet up precisely at the orbit of p. The composition $E_\alpha := \exp_N^\alpha \circ (\exp_S^\alpha)^{-1} : \partial D_S \to \partial D_N$ is an equivariant, orientation reversing diffeomorphism of their boundaries.

Now we are ready to define $\mathfrak{f}$, the smooth replacement for f. Let $\eta_N : [0,1] \to S^2$ be the straight line segment connecting 0 to $(\exp_N^\alpha)^{-1}(p) \in \partial D_N$. Define the curve $\beta_N := \exp_N^\alpha \circ \eta_N$. Define $\beta_S : [0,1] \to S^2$ similarly, except so that it goes in the opposite direction (from p to S). Finally let β be the concatenation of β_N and β_S in S^2. Define

$$\mathfrak{f}(\beta(t)) := \gamma(t),$$

and extend by G-equivariance to all of S^2 by requiring

$$\mathfrak{f}(\alpha_g(\beta(t))) = \lambda_g(\gamma(t))$$

.

Let $D^\pm$ be the closed upper (resp. lower) hemisphere of S^2. Let $D_{N_0} = (\exp_{N_0}^\lambda)^{-1}(D^+)$ and similarly for D_{S_0}. For any point outside of $\mathrm{Orb}_\alpha(p)$, say in the northern hemisphere, the map $\mathfrak{f}$ factors as

$$\mathfrak{f}(q) := \exp_N^\lambda \circ \hat{f} \circ (\exp_N^\alpha)^{-1}(q)$$

for some orientation preserving, equivariant identification $\hat{f}$ of D_N with the disk D_{N_0}, and similarly for the southern hemisphere. Thus the only place where non-differentiability could occur is at the equator.

Fortunately this is not a major issue. Gauss's Lemma (in Riemannian geometry, see [4, Lemma 3.5, p.69]) implies that β_N and β_S meet $\mathrm{Orb}_\alpha(p)$ orthogonally (as determined by the inner product $\langle\ ,\ \rangle^\alpha$), and this implies that β is a smooth arc, possibly after reparameterization[10].

Away from the poles, the action α is free. Let $S^2_* := S^2 \setminus \{N, S\}$. The Slice Theorem (see [15, Thm 5.7, p.40]) supplies a smooth map

$$\Psi_\alpha : S^2_* \to {S^2_*}/{G} \cong (0,1)$$

from the (twice-punctured) sphere to the quotient space under α, which we identify with (0,1). By composing with a diffeomorphism of $(0,1)$, it is clear that we have many options

[10]It may be necessary to reparameterize to ensure that the magnitude of the velocity is $\mathcal{C}^1$

for this identification. Since β is a smooth arc, we may choose an identification that satisfies:

$$\Psi_\alpha\big(\beta(t)\big) = t$$

If we take the product of this map Ψ_α with the map Φ_α supplied by the Smooth Orbit/Stabilizer Theorem, this produces a cylindrical coordinate chart on S^2_*:

$$\Phi_\alpha \times \Psi_\alpha : S^2_* \to S^1 \times (0,1)$$

Let $\iota : S^1 \hookrightarrow G$ be the inclusion of the circle subgroup into G. For all $x \in S^2_*$, the chart map above satisfies

$$\alpha\left(\iota \circ \Phi_\alpha(x), \beta \circ \Psi_\alpha(x)\right) = x.$$

A sybolic shuffling of our notations easily reveals that a factorization of $\mathfrak{f}$ when restricted to S^2_* is given by

$$\mathfrak{f} = \lambda \circ \big(\iota \circ \Phi_\alpha\big) \times \big(\gamma \circ \Psi_\alpha\big)$$

This shows that $\mathfrak{f}$ is also differentiable at the equator, and is thus differentiable everywhere on S^2.

□

5.3. **Proof of Theorem 2 if $H = S^1 \times \mathbb{Z}_2$, or $O(2)$, or $O(2) \times \mathbb{Z}_2$.**

For these conjugacy classes, the construction is similar to that of case 5.2, except that once we have specified β_N, $\mathfrak{f}$ is already completely determined.

It is worth mentioning that Gauss's Lemma is necessary once again to ensure that $\mathfrak{f}$ is smooth in neighborhoods of $\mathrm{Orb}_\alpha(p)$.

5.4. **Proof of Theorem 2 if H is a finite subgroup of $SO(3)$.**

If H is finite, then let Y_i be the H-equivariant i-skeleton of S^2 as described in Section 4. Using this, we will define $\mathfrak{f}$ through a three-step process:

(i) Define $\mathfrak{f}_0$ to agree with the restriction of f to $X_0 := f^{-1}(Y_0)$, and extend this to disk neighborhoods of the vertices of X_0 using the exponential map.

(ii) Use $f^{-1}(Y_1)$ to construct a 'suitably smooth' 1-skeleton X_1, and extend $\mathfrak{f}_0$ to $\mathfrak{f}_1$ by requiring it to take an equivariant tubular neighborhood of X_1 to that of Y_1.

(iii) Extend $\mathfrak{f}_1$ to the remaining fundamental domain, and extend to all of S^2 by equivariance to define $\mathfrak{f}$.

To begin, notice that H-CW decomposition of S^2 with respect to the standard action Λ is a G-CW decomposition with respect to the action λ. Define $X_0 := f^{-1}(Y_0)$. Let the set $P_0' := \{y_C\}_{C\in P_0}$ consist of the representative points listed in Section 4 (such as N, p, u_a etc.) whose orbits define the equivariant 0-cells, and let $x_C = f^{-1}(y_C)$ for each $y_C \in P_0'$. Define

$$U_0^r := \bigcup_{x\in X_0} \exp_x^\alpha(B_r(0)), \quad \text{and}$$

$$V_0^r := \bigcup_{y\in Y_0} \exp_y^\alpha(B_r(0)),$$

then choose an $\varepsilon > 0$ such that U_0^ε and V_0^ε are disjoint unions of disks.

Next, let the set $\{l_i\}_{i=1}^m =: P_1'$ $(m \leq 2)$ be an enumeration of the representative arcs from Section 4 whose orbits determine the equivariant 1-cells (*e.g.* $M, E_{1/2}, M_{1/2}^+$ etc.). Define σ_i and τ_i to be the source and terminus of the arc $f^{-1}(l_i)$. Careful inspection of the l_i for any orientation preserving group shows that it is possible to reorder the arcs so that $\tau_i = \sigma_{i+1}$ for each $i < m$. In other words it is possible to create a path from σ_1 to τ_m by concatenating the representative arcs l_i.

Let λ' and α' be the restrictions of the actions λ and α respectively to the subgroup $K := \mathrm{Stab}_\lambda(y_C) = \mathrm{Stab}_\alpha(x_C)$. Let

$$L_C : \left(T_{x_C}S^2, d\alpha'\right) \to \left(T_{y_C}S^2, d\lambda'\right)$$

be any equivariant, linear isometry of the tangent spaces that preserves orientation if $\deg(f) = 1$ and reverses orientation if $\deg(f) = -1$. These exist by Theorem 5. Finally, complete the first step by defining:

$$\mathfrak{f}_0 := \exp_{y_C}^\lambda \circ L_C \circ (\exp_{x_C}^\alpha)^{-1}\big|_{\exp_{x_C}^\alpha(B_{\varepsilon/2}(0))} \quad \text{for every} \quad y_C \in P_0',$$

and extending this by equivariance to all of $U_0^{\varepsilon/2}$.

Our current goal is to replace each continuous arc $f^{-1}(l_i)$ with a suitable smooth arc e_i. Choose some $\delta > 0$ small enough that

$$B_\delta(f^{-1}(l_i)) \cap B_\delta(f^{-1}(l_j)) \subseteq U_0^\varepsilon, \quad i \neq j.$$

Let e_1 be a smooth arc that:

(i) agrees with $\mathfrak{f}_0^{-1}(l_1)$ on $U_0^{\varepsilon/2}$,

(ii) is contained in $U_0^\varepsilon \cup B_\delta(f^{-1}(l_1))$, and

(iii) is transverse to every $\exp_{\sigma_1}^\alpha(\partial B_r(0))$ and $\exp_{\tau_1}^\alpha(\partial B_r(0))$ for $\varepsilon/2 \leq r \leq \varepsilon$.

These conditions merit some explanation. The first condition is forced upon us by the fact that we have already defined $\mathfrak{f}_0$ and we desire $\mathfrak{f}(e_1) = l_1$ for some extension $\mathfrak{f}$. The second condition guarantees that e_1 connects σ_1 to τ_1 in the same way that $f^{-1}(l_1)$ does, so that our resultant 1-skeleton will have the same combinatorial structure as Y_1. The third condition guarantees that $e_1 \cap \alpha_g(e_1) = \emptyset$ whenever $g \in G$ is non-trivial. All of these conditions can be guaranteed by combining techniques of elementary approximation theory. If G is cyclic, this is the only arc that we must replace, otherwise we must repeat this process once more:

Let e_2 be a smooth arc that:

(i) agrees with $\mathfrak{f}_0^{-1}(l_2)$ on $U_0^{\varepsilon/2}$,

(ii) is contained in $\left(U_0^\varepsilon \cup B_\delta(f^{-1}(l_2))\right) \setminus \left(\bigcup_{g\in G} \alpha_g(e_1)\right)$, and

(iii) is transverse to every $\exp_{\sigma_2}^\alpha(\partial B_r(0))$ and $\exp_{\tau_2}^\alpha(\partial B_r(0))$ for $\varepsilon/2 \leq r \leq \varepsilon$.

The alternate set in condition (ii) is clearly necessary, as our arcs should only intersect at their endpoints. Let Θ be the set mentioned in condition (ii). In order to know that such an e_2 can be constructed, it will suffice to show that $\mathfrak{f}_0^{-1}(l_2)$ lies in the same component of Θ as $f^{-1}(l_2) \cap \exp_{\sigma_2}^\alpha(\partial B_\varepsilon(0))$. If they weren't in the same component, then this would imply that $\det(L_C) = -\deg(f)$ which contradicts the construction of L_C, so such an e_2 exists.

We can now freely define $\mathfrak{f}_1(e_i) = l_i$. By the Slice Theorem, it is possible to construct α-equivariant[11] tubular neighborhoods $U(e_i) \subseteq S^2$ of the e_i which we may assume to be disjoint outside of $U_0^{\varepsilon/2}$. Similarly there are λ-equivariant tubular neighborhoods $V(l_i)$ of

[11]The astute reader will notice that the stabilizers of every 1-cell in each of the orientation preserving groups are all trivial. This means that the full power of the Slice Theorem is not necessary here, because

each l_i. If we set

$$U_1 := U_0 \cup \bigcup_{i=1}^{m} \bigcup_{g \in G} \alpha_g \left(U(e_i) \right) \quad \text{and}$$
$$V_1 := V_0 \cup \bigcup_{i=1}^{m} \bigcup_{g \in G} \lambda \left(V(l_i) \right),$$

then we can extend $\mathfrak{f}_1$ to an equivariant map $\mathfrak{f}_1 : \overline{U_1} \to \overline{V_1}$.

What remains is to extend the map to our fundamental domain. Let D_U be any connected component of $S^2 \setminus U_1$. There is only one component of $S^2 \setminus V_1$ that intersects $f(D)$. Let this region be called D_V. So far we have constructed a diffeomorphism $\mathfrak{f}_1\big|_{\partial} : \partial D_U \to \partial D_V$, and we would like to extend this map to a diffeomorphism of D_U to D_V.

Since the disk is null-homotopic, we can clearly extend $\mathfrak{f}_1\big|_{\partial}$ to a map $D_U \to D_V$. This map can be made to be a homeomorphism. There is a well defined obstruction theory for smoothing homeomorphisms of $\overline{D^n}$ that are diffeomorphisms on S^{n-1}, that was initially defined by Munkres in [11]. In an earlier paper [10] Munkres also showed that the groups in which such obstructions lie are trivial for dimensions ≤ 3. Thus we are able to extend $\mathfrak{f}_1$ to a diffeomorphism of D_U onto D_V.

Finally we extend this by equivariance to a diffeomorphism of $\mathfrak{f} : S^2 \to S^2$, which will be equivariant by construction.

5.5. **Proof of Theorem 2 if $H \leq O_3$ is a finite subgroup $\not\subseteq SO_3$.**

In this case, G may or may not contain (possibly multiple) copies of $\mathbb{Z}_2$ as a reflection. This implies that the fixed point sets of each reflection is a smoothly embedded copy of S^1. If there are no elements in G that correspond to a reflection across a plane, then the above procedure can be carried out just as before. If there are elements such as this, then the construction above can be carried out with a few caveats.

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there is no group action. However this general construction will be important for when the actions do not preserve orientation.

The embedded S^1s are a necessary part of the 1-skeleton, and cannot be altered. This is actually convenient, because it means that instead of replacing each $f^{-1}(l_i)$ with e_i, we can simply use the $f^{-1}(l_i)$ because they are already smooth arcs.

The proof in these cases relies on the full power of the Slice Theorem, which was not *strictly* necessary for the orientation preserving case. This is because now edges are allowed to have nontrivial stabilizers (reflections across themselves), and so the tubular neighborhoods must be *equivariant*.

This exhausts all possibilities for $[H] \in C_c(O_3)$ and hence concludes the proof of Theorem 2 in its entirety.

■

6. Proof of Corollary 3

In this section we drop the assumption of effectiveness, and address this possibility specifically. Following Palais [12], let $\mathcal{D} := \mathrm{Diff}(S^2)$ be the group of diffeomorphisms of S^2 under composition, and let $\mathcal{A} := \mathcal{A}(SO_3, S^2)$ be the space of continuous homomorphisms from $SO_3 \to \mathcal{D}$, where both are topologized with the compact-open topology. As in Section 2.1, this space can be naturally identified with the space of smooth actions of SO_3 on S^2. Since SO_3 is simple and compact, any homomorphism in $\mathcal{A}$ is either trivial or an embedding[12]. We only wish to consider effective actions, so let $\mathcal{E} := \{\tilde{\alpha} \in \mathcal{A} \mid \ker \tilde{\alpha} = 0\}$.

There is an action $\Upsilon : \mathcal{D} \times \mathcal{A} \to \mathcal{A}$ that is given by conjugation. That is to say, if $\tilde{\alpha} \in \mathcal{A}$ is the associated homomorphism of a smooth action $\alpha : SO_3 \times S^2 \to S^2$, $f \in \mathcal{D}$ and $g \in SO_3$, then Υ is defined by

$$\Upsilon(f, \tilde{\alpha})(g) := f \circ \alpha_g \circ f^{-1} = (\Upsilon_f \tilde{\alpha})_g$$

This action of $\mathcal{D}$ preserves the kernels of each element of $\mathcal{A}$, so it restricts to an action on $\mathcal{E}$. Theorem 2 shows that for all $\tilde{\alpha} \in \mathcal{E}$, there is some linear action $\tilde{\lambda} \in \mathcal{E}$, and some element $f \in \mathcal{D}$ that satisfies $\Upsilon_f \tilde{\alpha} = \tilde{\lambda}$. Conjugation by a change of basis matrix shows that there is no loss of generality in assuming that $\tilde{\lambda}$ is the standard inclusion of $SO_3 \hookrightarrow \mathcal{D}$. This shows

[12]Here by an embedding we mean both a homomorphism and a homeomorphism onto its image.

that all effective smooth actions $\tilde{\alpha}$ are in the same Υ-orbit as the standard inclusion, so the action Υ is transitive on $\mathcal{E}$. Clearly the trivial homomorphism $1 : g \mapsto \mathrm{id}_{S^2}$ is alone in its orbit, so we obtain the Υ-equivariant decomposition $\mathcal{A} = \mathcal{E} \sqcup \{1\}$.

If $f \in \mathrm{Stab}_\Upsilon(\tilde{\lambda})$, then $\Upsilon_f \tilde{\lambda} = \tilde{\lambda}$, or in other words,

$$f \circ \lambda_g \circ f^{-1} \; = \; \lambda_g, \quad \forall g \in SO_3 \qquad \text{or} \qquad f \circ \lambda_g \; = \; \lambda_g \circ f, \quad \forall g \in SO_3$$

Let $x \in S^2$, and let $g \in SO_3$ be such that λ_g is a non-trivial rotation about the axis determined by x. For any $f \in \mathrm{Stab}_\Upsilon(\tilde{\lambda})$, we have that

$$\lambda_g\big(f(x)\big) \; = \; f\big(\lambda_g x\big) = f(x)$$

$$\Longrightarrow$$

$$f(x) \; = \; \pm x$$

$$\Longrightarrow$$

$$f \; = \; \pm I \in SO_3 \leq \mathcal{D}$$

Hence we find that $\mathrm{Stab}_\Upsilon(\tilde{\lambda}) = \{\pm I\}$. Both SO_3 and S^2 are compact, so by [12, Cor. 2], we have that

$$\mathrm{Diff}_0(S^2) \; \cong \; {}^{\mathcal{D}}\!\big/_{\{\pm I\}} \; = \; {}^{\mathcal{D}}\!\big/_{\mathrm{Stab}_\Upsilon(\tilde{\lambda})} \; \cong \; \mathrm{Orb}_\Upsilon(\tilde{\lambda}) \; = \; \mathcal{E},$$

where $\mathrm{Diff}_0(S^2)$ is the subgroup of orientation preserving diffeomorphisms of the sphere. This proves the corollary.

$\square$

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