The Universal Trace

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Abstract

In these notes we calculate the coend

$$\int^{X\in\mathcal{C}} \mathcal{C}(X,X),$$

for a finite linear category $\mathcal{C}$, subject to a certain dimension restriction on indecomposable objects. In doing so, we explicitly describe the universal cowedge

$$\tau : \bigoplus_{X\in\mathcal{C}} \mathcal{C}(X,X) \to \int^{X\in\mathcal{C}} \mathcal{C}(X,X),$$

which we interpret as a kind of trace.

1 The Cowedge Condition

Here we give a specific interpretation of the idea of coends that is relevant to our situation. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $K : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor.

Definition A morphism of the form

$$\eta : \bigoplus_{X\in\mathcal{C}} K(X,X) \to D$$

is called a *cowedge* under K if for all $f : X \to Y$ the following square commutes:

$$\begin{array}{ccc} K(Y,X) & \xrightarrow{K(f,X)} & K(X,X) \\ {\scriptstyle K(Y,f)}\downarrow & & \downarrow{\scriptstyle \eta} \\ K(Y,Y) & \xrightarrow{\quad\eta\quad} & D \end{array}$$

In these notes, we will be concerned with the situtation where $\mathcal{C}$ is a finite $\mathbb{K}$-linear category, with $\mathbb{K}$ algebraically closed, $\mathcal{D} = \mathsf{Vec}$ and $K(\, , \,) = \mathcal{C}(\, , \,)$, the hom-functor. In this situation, $\mathcal{C}(f, X)$ is usually written as f^* and is precomposition with f, while $\mathcal{C}(Y, f)$ is usually written f_* and is postcompostion with f. By making the appropriate substitutions to the above diagram, we arrive at

$$\begin{array}{ccc} \mathcal{C}(Y,X) & \xrightarrow{\quad f^*\quad} & \mathcal{C}(X,X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \eta} \\ \mathcal{C}(Y,Y) & \xrightarrow{\quad\eta\quad} & V. \end{array}$$

Note that if $g \in \mathcal{C}(Y, X)$, then commutativity of the diagram implies $\eta(gf) = \eta(fg)$. This will be an important formula for us, so we give it a name.

Definition For the sake of these notes, a morphism

$$\eta : \bigoplus_{X \in \mathcal{C}} \mathcal{C}(X, X) \to V$$

is said to satisfy the *cowedge condition* if for any pair of morphisms f and g in $\mathcal{C}$, $\eta(gf) = \eta(fg)$ whenever both compositions are defined.

Thus we find that such a morphism out of the direct sum[1] is a cowedge under $\mathcal{C}(\, , \,)$ if and only if it satisfies the cowedge condition.

[1]Since finite linear categories are essentially small, up to equivalence this direct sum exists in Vec. Technically this fact is not necessary for our argument, but it allows our notation to be more economical.

Definition A *coend* is an initial cowedge. This means that τ is a coend of K if for every cowedge η under K there is a unique linear map ζ such that $\eta = \zeta\tau$. By abuse of notation the term coend also refers to the object at the codomain of the cowedge, and for this object we use the notation

$$\int^{X \in \mathcal{C}} K(X, X),$$

where the symbol X acts as an index or 'dummy variable' similar to the notation used in products and coproducts.

Let $\mathcal{I}$ be a set of representatives of isomorphism classes of indecomposable objects in $\mathcal{C}$. The conclusion of these notes is that, provided $\dim\big(\mathsf{End}(X)\big) \neq 0 \in \mathbb{K}$ for each $X \in \mathcal{I}$,

$$\int^{X \in \mathcal{C}} \mathcal{C}(X, X) \;\cong \mathbb{K}\mathcal{I}.$$

2 General Observations

Here we discuss several facts that will be important for our main result. First, we take advantage of the fact that our category $\mathcal{C}$ has an epi-monic factorization.

Lemma 2.1 (Fitting) *Let $f : X \to X$ be an endomorphism in $\mathcal{C}$. Then there exists an $N \in \mathbb{N}$ such that*

$$X \cong \ker(f^N) \oplus im(f^N).$$

Proof. We begin by factoring f via its image:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "e_1"'] & & X \\ & X_1 \arrow[ur, hook, "m_1"'] & \end{tikzcd}$$

Next, copy the inclusion of the image $m_1 : X_1 \hookrightarrow X$ to the left side of the diagram to form a parallelogram:

$$\begin{tikzcd} & X \arrow[rr, "f"] \arrow[dr, two heads, "e_1"] & & X \\ X_1 \arrow[ur, hook, "m_1"] \arrow[rr, "f_1"] & & X_1 \arrow[ur, hook, "m_1"] & \end{tikzcd}$$

This construction shows that given any endomorphism f, we can factor it as $f = m_1 e_1$, and from this we can form the endomorphism $f_1 = e_1 m_1 : X_1 \to X_1$ of the image of f. This creates a subobject $X_1 \subseteq X$, and we can repeat this process to obtain a chain of subobjects:

$$\cdots \subseteq X_4 \subseteq X_3 \subseteq X_2 \subseteq X_1 \subseteq X$$

Since all objects in $\mathcal{C}$ have finite length, this process must terminate in a minimal element after a finite number of steps

$$X_{N+1} = X_N \subseteq \cdots \subseteq X_4 \subseteq X_3 \subseteq X_2 \subseteq X_1 \subseteq X,$$

and this happens precisely when f_N is an isomorphism.

Armed with this information, we can patch the corresponding parallelograms together to form a 'quilt diagram':

$$\begin{tikzcd}
 & & & X \arrow[rr, "f"] \arrow[dr, two heads, "e_1"] & & X \arrow[rr, "f"] \arrow[dr, two heads, "e_1"] & & X \arrow[rr, dotted] \arrow[dr, dotted, two heads] & & X \\
 & & X_1 \arrow[ur, hook, "m_1"] \arrow[rr, "f_1"] \arrow[dr, two heads, "e_2"] & & X_1 \arrow[ur, hook, "m_1"] \arrow[rr, "f_1"] \arrow[dr, two heads, "e_2"] & & X_1 \arrow[ur, hook, "m_1"] \arrow[rr, dotted] \arrow[dr, dotted, two heads] & & X_1 \arrow[ur, hook, "m_1"] & \\
 & X_2 \arrow[ur, hook, "m_2"] \arrow[rr, "f_2"] \arrow[dr, dotted, two heads] & & X_2 \arrow[ur, hook, "m_2"] \arrow[rr, "f_2"] \arrow[dr, dotted, two heads] & & X_2 \arrow[ur, hook, "m_2"] \arrow[rr, dotted] \arrow[dr, dotted, two heads] & & X_2 \arrow[ur, hook, "m_2"] & & \\
X_N \arrow[ur, dotted, hook] \arrow[rr, "f_N"] & & X_N \arrow[ur, dotted, hook] \arrow[rr, "f_N"] & & X_N \arrow[ur, dotted, hook] \arrow[rr, dotted] & & X_N \arrow[ur, dotted, hook] & & &
\end{tikzcd}$$

Now define the following morphisms

$$\begin{aligned}
e &:= e_N e_{N-1} \cdots e_2 e_1 \\
m &:= m_1 m_2 \cdots m_{N-1} m_N \\
p &:= m f_N^{-N} e,
\end{aligned}$$

where the last formula makes sense since f_N is an isomorphism. Observe that p is an idempotent

$$\begin{aligned}
p^2 &= \left(m f_N^{-N} e\right)\left(m f_N^{-N} e\right) \\
&= m f_N^{-N} f_N^{N} f_N^{-N} e \\
&= m f_N^{-N} e \;=\; p,
\end{aligned}$$

and thus $X \cong \ker(p) \oplus \operatorname{im}(p)$. Finally notice that

$$\begin{aligned}\ker(p) &\cong \ker(e) \cong \ker(f^N)\,, \quad \text{and}\\ \operatorname{im}(p) &\cong \operatorname{im}(e) \cong \operatorname{im}(f^N).\end{aligned}$$

□

Note: This is an abstract formulation and proof of Fitting's Lemma, which was originally a statement about linear maps between finite dimensional vector spaces. For an exposition of the classical version, see *e.g.* [Lei15].

Corollary 2.1.1 *Any endomorphism $f : X \to X$ of an indecomposable object X in $\mathcal{C}$ is either an isomorphism or nilpotent.*

This corollary is a powerful insight into the endomorphism rings of indecomposables, and we will take dramatic advantage of it. The rest of the results in this section concern endomorphisms of a fixed indecomposable X.

Lemma 2.2 (Schur's Lemma for Indecomposables) *For any endomorphism $f : X \to X$, there is a unique scalar $\lambda \in \mathbb{K}$ such that $f - \lambda \cdot id_X$ is nilpotent.*

Proof. Let $L_f : \mathsf{End}(X) \to \mathsf{End}(X)$ be the map $g \mapsto fg$. This map L_f is a linear map between finite dimensional vector spaces over $\mathbb{K}$, so it necessarily has an eigenvector. Let $g \in \mathsf{End}(X)$ be an eigenvector of L_f with eigenvalue λ. This means that

$$\begin{aligned} fg \;=\; L_f g \;&=\; \lambda \cdot g\\ &\Longrightarrow\\ (f - \lambda \cdot \mathrm{id}_X)g \;&=\; 0.\end{aligned}$$

Since $g \neq 0$, $\operatorname{im}(g) \neq 0$ and hence $f - \lambda \cdot \mathrm{id}_X$ has a nontrivial kernel. By Corollary 2.1.1 we can conclude that $f - \lambda \cdot \mathrm{id}_X$ must be nilpotent.

For uniqueness, suppose that both $h := f - \lambda \cdot \mathrm{id}_X$ and $k := f - \mu \cdot \mathrm{id}_X$ are nilpotent. This would then require that

$$\begin{aligned} \lambda \cdot \mathrm{id}_X + h \;=\; f \;&=\; \mu \cdot \mathrm{id}_X + k\\ &\Longrightarrow\\ (\lambda - \mu) \cdot \mathrm{id}_X + h \;&=\; k.\end{aligned}$$

Notice that in the last equation above, the left-hand side is a polynomial in h. Any polynomial in a nilpotent variable is a unit precisely when the coefficient of 1 ($= \mathrm{id}_X$ in this case) is a unit in the field. Since the right-hand side is nilpotent, the left-had side must not be a unit, and hence $\lambda = \mu$. $\square$

Lemma 2.3 *For any indecomposable X, the identity id_X cannot be written as a sum of nilpotent morphisms.*

Proof. As an initial case, observe that

$$\begin{gathered} \mathrm{id}_X = h + k \\ \Longrightarrow \\ \mathrm{id}_X - h = k. \end{gathered}$$

We have already seen in the proof of Lemma 2.2 that this implies that h and k cannot both be nilpotent. We conclude that id_X is not the sum of two nilpotent morphisms.

Now suppose that in order to write id_X as a sum of nilpotents, it takes at least $n+1$ summands. This would show that

$$\begin{aligned} \mathrm{id}_X &= \sum_{i=0}^{n} h_i \\ \mathrm{id}_X - h_0 &= \sum_{i=1}^{n} h_i \\ \mathrm{id}_X &= (\mathrm{id}_X - h_0)^{-1} \cdot \sum_{i=1}^{n} h_i \\ \mathrm{id}_X &= \sum_{i=1}^{n} (\mathrm{id}_X - h_0)^{-1} h_i. \end{aligned}$$

On the right-hand side, each summand is a composition of an isomorphism and a nilpotent. Such compositions necessarily have nontrivial kernel, and hence must themselves be nilpotent by Corollary 2.1.1. This contradicts our assumption, since we have now written id_X as a sum of only n many nilpotents. Thus by induction, there can be no way of writing id_X as a sum of nilpotents. $\square$

Corollary 2.3.1 *Any finite $\mathbb{K}$-linear combination of nilpotents is again nilpotent.*

Proof. Let f be a finite $\mathbb{K}$-linear combination of nilpotents. By Proposition 2.2, there exists a unique λ such that $g := f - \lambda \cdot \mathrm{id}_X$ is nilpotent. Suppose for the sake of contradiction that $\lambda \neq 0$. This would imply that

$$\mathrm{id}_X \;=\; \frac{1}{\lambda}\big(f - g\big),$$

but the right-hand side is a finite sum of nilpotents, and this contradicts Proposition 2.3. $\lightning$ Thus we may conclude that $f = g$, so f is nilpotent as desired. $\square$

Proposition 2.4 *All morphisms in* $\mathsf{End}(X)$ *commute up to a nilpotent morphism.*

Proof. Let $u := f - \lambda \cdot \mathrm{id}_X$ and $v := g - \mu \cdot \mathrm{id}_X$ be nilpotent endomorphisms. Then we have that all commutators are equal to commutators of nilpotents, because

$$\begin{aligned} fg \;&=\; (\lambda \cdot \mathrm{id}_X + u)(\mu \cdot \mathrm{id}_X + v) \;=\; \lambda\mu \cdot \mathrm{id}_X + \lambda \cdot v + \mu \cdot u + uv \\ gf \;&=\; (\mu \cdot \mathrm{id}_X + v)(\lambda \cdot \mathrm{id}_X + u) \;=\; \lambda\mu \cdot \mathrm{id}_X + \mu \cdot u + \lambda \cdot v + vu \end{aligned}$$

$$\Longrightarrow$$

$$fg - gf \;=\; uv - vu.$$

Observe that the composition of two nilpotents necessarily has a nontrivial kernel, so by Corollary 2.1.1 it must again be nilpotent. Together with Corollary 2.3.1, this implies that all commutators of nilpotents are again nilpotent. Thus all commutators are nilpotent, and this is equivalent to the claim. $\square$

Definition Let X be any indecomposable object, 2.2 implies that there is a well-defined map

$$\lambda_X : \mathsf{End}(X) \to \mathbb{K},$$

which takes an endomorphism $f : X \to X$ and maps it to the unique scalar $\lambda_X(f)$ that makes $f - \lambda_X(f) \cdot \mathrm{id}_X$ nilpotent.

Proposition 2.5 *The map* λ_X *is linear, and for any two* $f, g : X \to X$,

$$\lambda_X(fg) = \lambda_X(gf).$$

Proof. Let $k \in \mathbb{K}$. Obersve that

$$(k \cdot f + g) - \big(k \cdot \lambda_X(f) + \lambda_X(g)\big) \cdot \mathrm{id}_X = k \cdot \big(f - \lambda_X(f) \cdot \mathrm{id}_X\big) + \big(g - \lambda_X(g) \cdot \mathrm{id}_X\big).$$

The right hand side is nilpotent by 2.3.1 and the definition of λ_X. Thus the left hand side is nilpotent, so by uniqueness, λ_X is linear.

By linearity and proposition corollary 2.4, we find that

$$\lambda_X(fg) - \lambda_X(gf) \;=\; \lambda_X(fg - gf) \;=\; 0,$$

and this proves the claim. □

After all of the above, the reader may have begun to suspect the following claim.

Proposition 2.6 *For every indecomposable X in $\mathcal{C}$ there exists a basis of* $\mathsf{End}(X)$*, where the matrix of (left multiplication by) every endomorphism is upper triangular and constant along the diagonal.*

Proof. The previous results of this section show that the set of all nilpotent morphisms $N \subsetneq \mathsf{End}(X)$ is an ideal, and in fact it is the Jacobson radical. By Nakayama's lemma, $N^2 \subsetneq N$ (provided $N \neq 0$), and there is some n such that $N^n = 0$. Thus we obtain a filtration

$$0 \subsetneq N^{n-1} \subsetneq \cdots \subsetneq N \subsetneq \mathsf{End}(X).$$

Any basis respecting this filtration will necessarily make all the matrices of left multiplication upper triangular. The statement about being constant along the diagonal follows from Lemma 2.2. □

The author suspected this for some time, and would like to thank David Speyer for his Math Overflow post [hes] which explained how the proof should go.

Note: The scalar map λ_X is easily seen to be the map that picks out the common scalar along the diagonal in this upper triangular basis.

Definition Let X be an indecomposable, and suppose $A \in \mathsf{End}\big(\bigoplus_i X\big)$ is given by its matrix coefficients $A = (A_{ij})_{ij}$, where each $A_{ij} : X \to X$. In analogy to the classical trace, define a map

$$T_X(A) := \sum_i \lambda_X(A_{ii})$$

Proposition 2.7 *Given two maps $A, B \in \mathsf{End}\big(\bigoplus_i X\big)$,*

$$T_X(AB) = T_X(BA)$$

Proof. This is a direct computation:

$$\begin{aligned}
T_X(AB) &= \sum_i \lambda_X\big((AB)_{ii}\big) \\
&= \sum_i \lambda_X \left(\sum_j A_{ij} B_{ji} \right) \\
&= \sum_{i,j} \lambda_X\big(A_{ij} B_{ji}\big) \\
&\overset{2.5}{=} \sum_{i,j} \lambda_X\big(B_{ji} A_{ij}\big) \\
&= \sum_j \lambda_X \left(\sum_i B_{ji} A_{ij} \right) \\
&= \sum_j \lambda_X\big((BA)_{jj}\big) \\
&= T_X(BA).
\end{aligned}$$

□

Lemma 2.8 *Suppose $X \not\cong Z$ are each indecomposable and $f : X \to X$ factors through Z like*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g"'] & & X \\
& Z \arrow[ur, "h"'] &
\end{tikzcd},$$

then f is nilpotent.

Proof. Suppose for the sake of contradiction that f is an isomorphism. This would imply that $p := gf^{-1}h : Z \to Z$ is an idempotent. This would then imply that $Z \cong X \oplus X'$, and hence $Z \cong X$ by the indecomposability of Z. Thus f must not have been an isomorphism, and therefore by 2.1.1 it must be nilpotent. □

3 Consequences of the Cowedge Condition

For this section, let

$$\eta : \bigoplus_{X \in \mathcal{C}} \mathcal{C}(X, X) \to V$$

be a cowedge under $\mathcal{C}(\ ,\ )$.

Lemma 3.1 *If $f : X \to X$ is a nilpotent endomorphism, then $\eta(f) = 0$.*

Proof. Here we reuse the factorization argument from 2.1, so the reader is encouraged to look back at the quilt 2. Let $N \in \mathbb{N}$ be the smallest such that $f^N = 0$. Note that $\operatorname{im}(f^{n+1}) \subsetneq \operatorname{im}(f^n)$ for every $n \leq N-1$. Thus our N (the degree of nilpotency) is precisely the terminating N at the bottom of the quilt. In particular this means that $f_N = 0$ since it is an endomorphism of $X_N = 0$. Using the cowedge condition, we find that

$$\begin{aligned}
\eta(f) &= \eta\big(m_1 e_1\big) \\
&= \eta\big(e_1 m_1\big) \\
&= \eta\big(f_1\big) \\
&= \eta\big(m_2 e_2\big) \\
&= \eta\big(e_2 m_2\big) \\
&= \eta\big(f_2\big) \\
&\ \ \vdots \\
&= \eta\big(f_N\big) \\
&= \eta(0) \ = \ 0.
\end{aligned}$$

□

Corollary 3.1.1 *If f is an endomorphism of an indecomposable X, then $\eta(f) = \lambda_X(f) \cdot \eta(id_X)$ for some unique $\lambda \in \mathbb{K}$.*

Proof. Combine lemma 3.1 with the definition of λ_X. □

Lemma 3.2 *Suppose $\phi : Y \to \bigoplus_{i=1}^n X$ is an isomorphism, with X indecomposable (Y is X-isotypic). Then for any endomorphism f of Y, we have that*

$$\eta(f) \ = \ T_X\big(\phi f \phi^{-1}\big) \cdot \eta(id_X).$$

Moreover, this scalar $T_X\big(\phi f \phi^{-1}\big)$ *does not depend on the chosen isomorphism* ϕ*.*

Proof. Conjugating f by ϕ, we obtain an endomorphism $\phi f \phi^{-1} : \bigoplus_i X \to \bigoplus_i X$. We can write $\phi f \phi^{-1}$ as a sum of matrix coefficients like

$$\begin{aligned}
\phi f \phi^{-1} &= \mathrm{id}_Y \phi f \phi^{-1} \mathrm{id}_Y \\
&= \left(\sum_i \iota_i \pi_i\right) \phi f \phi^{-1} \left(\sum_j \iota_j \pi_j\right) \\
&= \sum_{i,j} \iota_i (\pi_i \phi f \phi^{-1} \iota_j) \pi_j .
\end{aligned}$$

Now we can apply η to obtain

$$\begin{aligned}
\eta(f) &= \eta(f \phi^{-1} \phi) \\
&= \eta(\phi f \phi^{-1}) \\
&= \eta\left(\sum_{i,j} \iota_i (\pi_i \phi f \phi^{-1} \iota_j) \pi_j\right) \\
&= \sum_{i,j} \eta\Big(\iota_i (\pi_i \phi f \phi^{-1} \iota_j) \pi_j\Big) \\
&= \sum_{i,j} \eta\Big((\pi_i \phi f \phi^{-1} \iota_j) \pi_j \iota_i\Big) \\
&= \sum_i \eta\big(\pi_i \phi f \phi^{-1} \iota_i\big) \\
&\overset{3.1.1}{=} \sum_i \lambda_X\big(\pi_i \phi f \phi^{-1} \iota_i\big) \cdot \eta(\mathrm{id}_X) \\
&= \left(\sum_i \lambda_X\big(\pi_i \phi f \phi^{-1} \iota_i\big)\right) \cdot \eta(\mathrm{id}_X) \\
&= T_X\big(\phi f \phi^{-1}\big) \cdot \eta(\mathrm{id}_X).
\end{aligned}$$

This establishes the formula. To see that the map ϕ is irrelevant, let ψ be another such isomorphism. A quick computation shows that

$$\begin{aligned}
T_X\big(\phi f \phi^{-1}\big) &= T_x\big(\phi f \psi^{-1} \psi \phi^{-1}\big) \\
&\overset{2.7}{=} T_x\big(\psi \phi^{-1} \phi f \psi^{-1}\big) \\
&= T_x\big(\psi f \psi^{-1}\big).
\end{aligned}$$

$\square$

Note: In light of 3.2, we will write $T_X(f) := T_X\big(\phi f \phi^{-1}\big)$. This is not only easier to read, but also avoids introducing dummy variables that don't effect the output.

Proposition 3.3 *Let $f : Y \to Y$ be an endomorphism of an arbitrary object in $\mathcal{C}$. Then*

$$\eta(f) = \sum_{X \in \mathcal{I}} T_X(\pi_X f \iota_X) \cdot \eta(id_X).$$

Proof. Every object in $\mathcal{C}$ has finite length, and so the general Krull-Schmidt theorem (see for *e.g.* [EGNO15]) holds. Let us choose an isomorphism

$$Y \cong \bigoplus_{X \in \mathcal{I}} \left(\bigoplus_{i_X = 1}^{n_X} X \right)$$

and define the X-isotypic components to be the summands

$$Y_X \cong \bigoplus_{i_X = 1}^{n_X} X. \tag{1}$$

As before, we can rewrite f as a sum like

$$f = \sum_{X, Z \in \mathcal{I}} \iota_X \pi_X f \iota_Z \pi_Z.$$

Then applying η, we find

$$\begin{aligned}
\eta(f) &= \sum_{X, Z \in \mathcal{I}} \eta\big(\iota_X \pi_X f \iota_Z \pi_Z\big) \\
&= \sum_{X, Z \in \mathcal{I}} \eta\big(\pi_X f \iota_Z \pi_Z \iota_X\big) \\
&\overset{2.8}{=} \sum_{X \in \mathcal{I}} \eta\big(\pi_X f \iota_X\big) \\
&\overset{3.2}{=} \sum_{X \in \mathcal{I}} T_X(\pi_X f \iota_X) \cdot \eta\big(\mathrm{id}_X\big) \\
&=: \sum_{X \in \mathcal{I}} T_X(\pi_X f \iota_X) \cdot \eta(\mathrm{id}_X).
\end{aligned}$$

$\square$

Heuristically, this last proposition says that any cowedge under $\mathcal{C}(\,,\,)$ is completely determined by its values on the set $\{\mathrm{id}_X\}_{X\in\mathcal{I}}$. This fact will be crucial to proving that $\mathbb{K}\mathcal{I}$ has the specified universal property.

4 The Universal Cowedge

We begin with a definition:

Definition The map τ is determined by the formula

$$\begin{aligned}\tau : \bigoplus_{Y\in\mathcal{C}} \mathcal{C}(Y,Y) &\longrightarrow \mathbb{K}\mathcal{I}\\ \left(Y \xrightarrow{f} Y\right) &\longmapsto \sum_{X\in\mathcal{I}} T_X(\pi_x f \iota_X)\cdot X.\end{aligned}$$

Note: After reading this definition, it should be noted that although we have written down this formula with the sole purpose of constructing a cowedge, Proposition 3.3 only furnishes a necessary condition. We wish to show that this formula is sufficient.

Proposition 4.1 *Let Y and Y' be two arbitrary objects in $\mathcal{C}$, and let $f : Y \to Y'$ and $g : Y' \to Y$ be any morphisms between them. Then $\tau(fg) = \tau(gf)$,* i.e. *τ is a cowedge.*

Proof.

$$
\begin{aligned}
\tau(fg) &:= \sum_{X\in\mathcal{I}} T_X(\pi_X fg\iota_X)\cdot X \\
&= \sum_{X\in\mathcal{I}} T_X\left(\pi_X f\left(\sum_{Z\in\mathcal{I}}\iota_Z\pi_Z\right)g\iota_X\right)\cdot X \\
&\overset{2.8}{=} \sum_{X\in\mathcal{I}} T_X\big(\pi_X f\iota_X\pi_X g\iota_X\big)\cdot X \\
&= \sum_{X\in\mathcal{I}} T_X\big((\pi_X f\iota_X)(\pi_X g\iota_X)\big)\cdot X \\
&\overset{2.7}{=} \sum_{X\in\mathcal{I}} T_X\big((\pi_X g\iota_X)(\pi_X f\iota_X)\big)\cdot X \\
&\overset{2.8}{=} \sum_{X\in\mathcal{I}} T_X\left(\pi_X g\left(\sum_{Z\in\mathcal{I}}\iota_Z\pi_Z\right)f\iota_X\right)\cdot X \\
&= \sum_{X\in\mathcal{I}} T_X\big(\pi_X gf\iota_X\big)\cdot X \\
&= \tau(gf).
\end{aligned}
$$

□

We have now done enough legwork so that the proof of our main theorem is easy to compile.

Theorem 4.2 *The cowedge*

$$\tau : \bigoplus_{X\in\mathcal{C}}\mathcal{C}(X,X)\longrightarrow \mathbb{K}\mathcal{I},$$

is initial. In particular, we obtain a canonical identification

$$\mathbb{K}\mathcal{I}\cong\int^{X\in\mathcal{C}}\mathcal{C}(X,X).$$

Proof. Let

$$\eta : \bigoplus_{X\in\mathcal{C}}\mathcal{C}(X,X)\to V$$

be an arbitrary cowedge. By Proposition 3.3, the value of η is determined by the values $\{\eta(\mathrm{id}_X)\}_{X\in\mathcal{I}}$. Thus we can define a map

$$\begin{aligned} \zeta : \mathbb{K}\mathcal{I} &\longrightarrow V \\ X &\longmapsto \eta(\mathrm{id}_X). \end{aligned}$$

From the construction, it is clear that $\eta = \zeta\tau$. Now suppose

$$\zeta' : \mathbb{K}\mathcal{I} \longrightarrow V,$$

is any linear map that satisfies $\eta = \zeta'\tau$. Then it is immediate that

$$\begin{gathered} \eta(\mathrm{id}_X) \;=\; \zeta'\big(\tau(\mathrm{id}_X)\big) \;=\; \zeta'(X) \\ \Longrightarrow \\ \zeta' = \zeta, \end{gathered}$$

so this factorization is unique. $\square$

5 Multiplicative Structure

In this section we consider the case when $\mathcal{C}$ is not just finite, $\mathbb{K}$-linear abelian, but also monoidal. [This section should have more exposition and references here].

The conclusion of this section is that when $\mathcal{C}$ is a finite tensor category, $\mathbb{K}\mathcal{I}$ has a natural structure of a $\mathbb{K}$-algebra (aka the Green Ring), and that τ is suitably compatible with this structure. This is none other than the representation ring, tensored over $\mathbb{Z}$ with the $\mathbb{K}$. In various other works (see *e.g.* [CVOZ14]) this ring is denoted by the dissapointingly nondescript $r(\mathcal{C})$. Green himself [Gre64] used the notation $a(G)$, when $\mathcal{C} = \mathsf{Rep}_{\mathbb{K}}(G)$. We prefer our notation, but are open to suggestions.

Proposition 5.1 *The kernel of τ is an ideal with respect to the tensor product, in the sense that*

$$\tau(k) = 0 \qquad \Longrightarrow \qquad \forall f\,,\ \tau\big(f \otimes k\big) = 0.$$

Note: This should not be confused with the notion of 'tensor ideal' (*c.f.* [EO18]) since τ does not respect composition in any meaningful way.

Proof.

$$\begin{aligned}\tau\Big(f\otimes(gh-hg)\Big) &= \tau\Big(f\otimes gh - f\otimes hg)\Big)\\ &= \tau\big((f\otimes g)(\mathrm{id}\otimes h)-(\mathrm{id}\otimes h)(f\otimes g))\big)\\ &= \ 0\end{aligned}$$

□

This can be used to give the following

Definition The product on $\mathbb{K}\mathcal{I}$ is determined by the formula

$$\tau(f)\cdot\tau(g) := \tau\big(f\otimes g\big).$$

Since τ is surjective, this product is everywhere-defined, and Proposition 5.1 shows that the product is well-defined. Next, observe that

$$\begin{aligned}\tau\big(f\oplus g\big) &= \tau\big(\iota_1 f\pi_1+\iota_2 g\pi_2\big)\\ &= \tau\big(\iota_1 f\pi_1\big)+\tau\big(\iota_2 g\pi_2\big)\\ &= \tau\big(\pi_1\iota_1 f\big)+\tau\big(\pi_2\iota_2 g\big)\\ &= \tau\big(f\big)+\tau\big(g\big).\end{aligned}$$

In particular, this shows that for any $X,Y\in\mathbb{K}\mathcal{I}$, we have

$$\begin{aligned}X+Y &= \tau\big(\mathrm{id}_X\big)+\tau\big(\mathrm{id}_Y\big)\\ &= \tau\big(\mathrm{id}_X\oplus\mathrm{id}_Y\big)\\ &= \tau\big(\mathrm{id}_{X\oplus Y}\big),\ \ \text{and}\\ X\cdot Y &= \tau\big(\mathrm{id}_X\otimes\mathrm{id}_Y\big)\\ &= \tau\big(\mathrm{id}_{X\otimes Y}\big).\end{aligned}$$

The above calculation shows that the ring structure we have defined on $\mathbb{K}\mathcal{I}$ using abstract endomorphisms matches the ring structure obtained by thinking of $\mathbb{K}\mathcal{I}$ as the Green ring of $\mathcal{C}$. This compatibility of τ with the tensor product serves to solidify the interpretation of τ as a kind of trace. To illustrate this, let us do some generalized character theory.

Let G be a finite group. There is a category $\mathsf{Rep}_{\mathcal{C}}(G)$ of $\mathcal{C}$-valued G-representations, whose objects are pairs $\big(V,\rho\big)$, where $V\in\mathcal{C}$ and $\rho: G\to$

$\mathsf{End}(V)$. The morphisms are those morphisms in $\mathcal{C}$ that are G-equivariant (*a.k.a.* intertwiners), and composition is simply composition in $\mathcal{C}$. If $\mathcal{B}G$ is the one object category whose morphisms are G, then $\mathsf{Rep}_{\mathcal{C}}(G)$ is isomorphic to the functor category $\mathcal{C}at\big(\mathcal{B}G, \mathcal{C}\big)$.

Definition The character χ_ρ of a representation $\big(V, \rho\big)$ is the function

$$\begin{aligned} \chi_\rho : G &\longrightarrow \mathbb{K}\mathcal{I} \\ g &\longmapsto \tau\big(\rho(g)\big). \end{aligned}$$

As with classical representation theory, the properties of τ imply that the map

$$\begin{aligned} \chi : \int^{(V,\rho)} \mathsf{Rep}_{\mathcal{C}}(G)\big((V,\rho)\,,\,(V,\rho)\big) &\longrightarrow \mathsf{Set}\left(G\,,\,\int^{X} \mathcal{C}(X,X)\right) \\ (V,\rho) &\longmapsto \chi_\rho \end{aligned}$$

is a ring homomorphism. In the above we have applied Theorem 4.2 twice to identify the left-hand side with the Green ring of the category $\mathsf{Rep}_{\mathcal{C}}(G)$ (the representation ring), and the right-hand side with the ring of functions from G to the Green ring of $\mathcal{C}$.

6 Applications to Pivotal Tensor Categories

Definition A *tensor* category (over $\mathbb{K}$) is $\mathbb{K}$-linear, abelian, rigid, monoidal category, where $\mathsf{End}(\mathbb{1}) \cong \mathbb{K}$.

A particularly nice class of tensor categories is that of *fusion* categories.

Definition A *fusion* category is a finite, simisimple tensor category.

Example 6.1 The following are fusion:

- the category Vec of vector spaces over $\mathbb{K}$,
- the category $\mathsf{Vec}(G)$ of G-graded vector spaces, where G is a finite group.
- the $\mathsf{Rep}_{\mathbb{K}}(G)$, representations of a finite group G, provided $|G| \neq 0 \in \mathbb{K}$, and more generally

- the category $\mathsf{Rep}_{\mathbb{K}}(H)$, representations of a finite dimensional, semisimple Hopf algebra.

Definition A *pivotal structure* on a rigid monoidal category is a natural isomorphism $a : \mathrm{id}_{\mathcal{C}} \to (-)^{**}$ that is monoidal, *i.e.* for all $X, Y \in \mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd} & X \otimes Y \arrow[dl, "a_{X\otimes Y}"'] \arrow[dr, "a_X \otimes a_Y"] & \\ (X \otimes Y)^{**} \arrow[rr, "\cong"] & & X^{**} \otimes Y^{**} \end{tikzcd}$$

A category is called *pivotal* is it is equipped with a preferred pivotal structure.

For this section, let $\mathcal{C}$ be a pivotal fusion category. In [Shi17] and [Shi19], Shimizu defines the algebra $\mathsf{CF}(\mathcal{C})$ of class functions. In our notation, we have that

Definition As a vector space,

$$\mathsf{CF}(\mathcal{C}) := \mathcal{C}\left(\mathbb{1}\ ,\ \int^{X} X^* \otimes X\right).$$

Shimizu goes on to show that $\mathsf{CF}(\mathcal{C})$ has connections with the Grothendieck ring $\mathsf{Gr}(\mathcal{C})$, and can be used to determine whether a braiding is nondegenerate. Using the pivotal structure, we can define a morphism $k_X : \mathcal{C}(X, X) \to \mathsf{CF}(\mathcal{C})$ for every $X \in \mathcal{C}$ by the composition:

$$\mathcal{C}(X, X) \to \mathcal{C}(\mathbb{1}, {}^*X \otimes X) \to \mathcal{C}(\mathbb{1}, X^* \otimes X) \to \mathcal{C}\left(\mathbb{1}\ ,\ \int^{X} X^* \otimes X\right) =: \mathsf{CF}(\mathcal{C}).$$

By the naturality of the pivotal structure, it is easily seen that for a given pair $f : Y \to X$ and $g : X \to Y$, we have $k_X(fg) = k_Y(gf)$. In other words, these maps combine to form a cowedge k under $\mathcal{C}(\ ,\ )$. By the universal property of $\mathbb{K}\mathcal{I}$, we obtain a unique map

$$\kappa : \mathbb{K}\mathcal{I} \to \mathsf{CF}(\mathcal{C}),$$

such that $k = \kappa\tau$.

In the setting of tensor categories, semisimplicity is equivalent to the exactness of the functor $\mathcal{C}(\mathbb{1}, -)$. This exactness condition immediately implies that κ is the following isomorphism:

$$
\begin{aligned}
\mathbb{K}\mathcal{I} &= \int^X \mathcal{C}(X, X) \\
&\cong \int^X \mathcal{C}(\mathbb{1}, {}^*X \otimes X) \\
&\cong \int^X \mathcal{C}(\mathbb{1}, X^* \otimes X) \\
&\cong \mathcal{C}\left(\mathbb{1}\ ,\ \int^X X^* \otimes X\right) \quad = \quad \mathsf{CF}(\mathcal{C}).
\end{aligned}
$$

Furthermore, semisimplicity implies that the Green ring is the same as the Grothendieck ring. These casual observations imply that

Corollary 6.0.1 *For any pivotal fusion category, the map*

$$\kappa : \mathsf{Gr}(\mathcal{C}) \longrightarrow \mathsf{CF}(\mathcal{C})$$

is an isomorphism.

7 Categorical Traces

We now proceed to investigate a categorified version of the coend construction. We will find that there is a notion of the trace of a functor, and that the coend we have investigated is only the trace of the identity functor. In this section we will make considerable use of coend calculus. For background on these techniques, see *e.g.* [Loregian].

Let $\mathcal{C}$ be any finite $\mathbb{K}$-linear category, and let $F : \mathcal{C} \to \mathcal{C}$ be any $\mathbb{K}$-linear functor.

Definition The trace $T(F)$ of F is defined to be the vector space:

$$T(F) := \int^{X \in \mathcal{C}} \mathcal{C}(X, FX)$$

Note: This construction is guaranteed to exist by the finiteness of $\mathcal{C}$, and the use of a universal property provides a vector space which is well-defined up to an isomorphism which is unique.

Proposition 7.1 *Let ι^H denote the cowedge corresponding to $T(H)$. Given two functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$, there is a canonical isomorphism*

$$\begin{gathered} T(FG) \xrightarrow{\zeta_{F,G}} T(GF) \\ \iota^{FG}_X\Big(X \xrightarrow{f} FGX\Big) \overset{\zeta_{F,G}}{\longmapsto} \iota^{GF}_{GX}\Big(GX \xrightarrow{Gf} GFGX\Big). \end{gathered}$$

Proof. This is a direct computation using coend calculus and liberal use of the Yoneda lemma:

$$\begin{aligned}
\mathsf{Vec}\left(\int^X \mathcal{C}(X, FGX)\,,\, V\right) &\cong \int_X \mathsf{Vec}\Big(\mathcal{C}(X\,,\, FGX)\,,\, V\Big) \\
&\overset{よ}{\cong} \int_X \mathrm{Nat}\bigg(\mathcal{D}(-, GX)\,,\, \mathsf{Vec}\Big(\mathcal{C}(X, F-)\,,\, V\Big)\bigg) \\
&\cong \int_X \int_Y \mathsf{Vec}\bigg(\mathcal{D}(Y, GX)\,,\, \mathsf{Vec}\Big(\mathcal{C}(X, FY)\,,\, V\Big)\bigg) \\
&\cong \int_X \int_Y \mathsf{Vec}\Big(\mathcal{D}(Y, GX) \otimes \mathcal{C}(X, FY)\,,\, V\Big) \\
&\cong \int_Y \int_X \mathsf{Vec}\Big(\mathcal{C}(X, FY) \otimes \mathcal{D}(Y, GX)\,,\, V\Big) \\
&\cong \int_Y \int_X \mathsf{Vec}\bigg(\mathcal{C}(X, FY)\,,\, \mathsf{Vec}\Big(\mathcal{D}(Y, GX)\,,\, V\Big)\bigg) \\
&\cong \int_Y \mathrm{Nat}\bigg(\mathcal{C}(-, FY)\,,\, \mathsf{Vec}\Big(\mathcal{D}(Y, G-)\,,\, V\Big)\bigg) \\
&\overset{よ}{\cong} \int_Y \mathsf{Vec}\Big(\mathcal{D}(Y\,,\, GFY)\,,\, V\Big) \\
&\cong \mathsf{Vec}\left(\int^Y \mathcal{D}(Y\,,\, GFY)\,,\, V\right) \\
&\overset{よ}{\Longrightarrow}
\end{aligned}$$

$$\int^X \mathcal{C}(X, FGX) \cong \int^Y \mathcal{D}(Y, GFY).$$

From the chain of isomorphisms above, the Yoneda lemma supplies an explicit isomorphism of vector spaces. Chasing through the above computation on the level of elements yields the desired map. □

If $\mathcal{C}$ is a fusion category over $\mathbb{K}$, a $\mathcal{C}$-module category is a pair $(\mathcal{M}, \rho)$, where $\mathcal{M}$ is a $\mathbb{K}$-linear category, and $\rho = (\rho^0, \rho^1) : \mathcal{C} \to \mathsf{End}(\mathcal{M})$ is a monoidal functor. This is a categorical analogue of a module over a ring. For $X \in \mathcal{C}$ and $M \in \mathcal{M}$, often we adopt the shorthand $X \triangleright_\rho M := \rho^0(X)(M)$, or simply $X \triangleright M$ when the functor ρ is clear from context. Given two module categories $(\mathcal{M}, \rho)$ and $(\mathcal{N}, \nu)$, we can define the direct sum

$$(\mathcal{M}, \rho) \boxplus (\mathcal{N}, \nu) := \big(\mathcal{M} \boxplus \mathcal{N}, X \triangleright_{\rho \boxplus \nu} (M, N) := (X \triangleright_\rho M, X \triangleright_\nu N)\big)$$

We say that $(\mathcal{M}, \rho)$ is indecomposable if it is not equivalent to a direct sum of nontrivial module categories. For example, the indecomposable module categories for $\mathsf{Vec}(G)$ correspond to transitive G-sets, together with data encoding the tensor structure maps ρ^1. The transitive G-sets are furthermore in bijection with conjugacy classes of subgroups $L \leq G$, and the extra data is encoded by certain cohomology classes in $H^2(L; \mathbb{K}^\times)$. This classification is due to Ostrik in [module cats over the drinfeld...].

Given two module categories $(\mathcal{M}, \rho)$ and $(\mathcal{N}, \nu)$ over $\mathsf{Vec}(G)$, we can define their Deligne tensor product by the formula

$$g \triangleright_{\rho \boxtimes \nu} (M \boxtimes N) := (g \triangleright_\rho M) \boxtimes (g \triangleright_\nu N).$$

It then makes sense to ask how the tensor product of two indecomposable module categories decomposes into indecomposables. In the classical case, the question of how tensor products of irreducibles decompose into irreducibles is the data needed to compute the Brauer ring $\Omega(G)$. This higher representation theory admits its own version of a Brauer ring, now with indecomposable module categories replacing irreducible representations.

Classically, a key tool for these types of computations is the character associated to a representation (V, ρ). This is simply defined to be the trace of the ring homomorphism ρ. Continuing our analogy, we establish a new

Definition Given a module category $\rho : \mathsf{Vec}(G) \to \mathsf{End}(\mathcal{M})$, the character of (M, ρ) is the assignment

$$\begin{aligned} \chi_\rho : G &\longrightarrow \mathsf{Vec} \\ g &\longmapsto T\big(\rho(g)\big). \end{aligned}$$

Proposition 7.2 *The following properties hold for χ:*

1. $\chi_{\rho \boxplus \nu}(g) \cong \chi_\rho(g) \oplus \chi_\nu(g)$
2. $\chi_{\rho \boxtimes \nu}(g) \cong \chi_\rho(g) \otimes \chi_\nu(g)$

Proof. The proof is analogous to the classical case, and follows from the definitions by using some coend calculus. □

Now consider the vector spaces determined by χ_ρ for some $(\mathcal{M}, \rho)$ a module category for $\mathsf{Vec}(G)$. Note that the definition via universal construction did not take advantage of the tensor structure maps of ρ. Since different choices of ρ^1 for the tensor structure of ρ can yield inequivalent module categories, the object $\chi_\rho(a)$ as a mere vector space cannot distinguish certain modules from one another. We would like to use the extra data of the tensor structure to upgrade our vector spaces, in the hopes of allowing χ_ρ to be a complete invariant. This leads us to the following

Proposition 7.3 *For every $g, a \in G$, the universal property of χ_ρ determines a map $\alpha(g,a) : \chi_\rho(a) \to \chi_\rho(gag^{-1})$, and for any third $h \in G$, this map satisfies,*

$$\alpha(h, gag^{-1}) \circ \alpha(g,a) = \alpha(hg, a)$$

Proof. Let $M \in \mathcal{M}$ and $f : M \to a \triangleright M$. Consider the following construction

$$\phi_g : f \mapsto \big((\rho^1_{g,ag^{-1}})^{-1} \triangleright (g \triangleright M)\big) \circ \big(g \triangleright ((\rho^1_{ag^{-1},g}) \triangleright M)\big) \circ (g \triangleright f)$$

Using this, let us define a map $\mathcal{M}(M, a \triangleright M) \to \chi_\rho(gag^{-1})$, by the formula

$$\Phi_g : f \mapsto \iota^{\rho(gag^{-1})}_{(g \triangleright M)}\big(\phi_g(f)\big).$$

Now suppose f factors as $j \circ k$, where $k : M \to N$ and $j : N \to a \triangleright M$. Since $g \triangleright - : \mathcal{M} \to \mathcal{M}$ is a functor, we find that

$$\begin{aligned}
\Phi_g(j \circ k) &= \iota^{\rho(gag^{-1})}_{(g \triangleright M)}\big(\phi_g(j \circ k)\big) \\
&= \iota^{\rho(gag^{-1})}_{(g \triangleright M)}\big(\phi_g(j) \circ (g \triangleright k)\big) \\
&= \iota^{\rho(gag^{-1})}_{(g \triangleright N)}\Big(\big(gag^{-1} \triangleright (g \triangleright k)\big) \circ \phi_g(j)\Big) \\
&= \iota^{\rho(gag^{-1})}_{(g \triangleright N)}\Big(\phi_g\big((a \triangleright k) \circ j\big)\Big) \\
&= \Phi_g\big((a \triangleright k) \circ j\big).
\end{aligned}$$

This computation shows that Φ_g satisfies the cowedge condition for $\mathcal{C}(-, a\triangleright =)$, and thus by the universal property of $\chi_\rho(a)$, determines a unique map $\alpha(g,a)$ from $\chi_\rho(a)$ to $\chi_\rho(gag^{-1})$ such that $\Phi_g = \alpha(g,a) \circ \iota^{\rho(a)}$.

Finally, consider the map $\Phi_h \circ \phi_g = \iota_N^{\rho(hgag^{-1}h^{-1})} \circ \phi_h \circ \phi_g$

$$
\begin{tikzcd}
\mathcal{M}(M, a \triangleright M) \arrow[rr, "\phi_{hg}"] \arrow[dr, "\iota_M^{\rho(a)}"] \arrow[dddd, "\phi_g"] & & \mathcal{M}\big(M, (hgag^{-1}h^{-1}) \triangleright (hg \triangleright M)\big) \arrow[dd, "\iota_{hg \triangleright M}^{\rho(hgag^{-1}h^{-1})}"] \\
& \chi_\rho(a) \arrow[dr, dashed, "\alpha(hg,a)"] \arrow[dd, dashed, "\alpha(g,a)"] & \\
& & \chi_\rho\big(hgag^{-1}h^{-1}\big) \\
& \chi_\rho\big(gag^{-1}\big) \arrow[ur, dashed, "\alpha(h,gag^{-1})"] & \\
\mathcal{M}\big(M, (gag^{-1}) \triangleright (g \triangleright M)\big) \arrow[ur, dashed, "\iota_{g \triangleright M}^{\rho(gag^{-1})}"] \arrow[rr, "\phi_h"] & & \mathcal{M}\Big(M, (hgag^{-1}h^{-1}) \triangleright \big(h \triangleright (g \triangleright M)\big)\Big) \arrow[uu, "\iota_{h \triangleright (g \triangleright M)}^{\rho(hgag^{-1}h^{-1})}"]
\end{tikzcd}
$$

For the sake of legibility we will suppress the superscripts from the ι maps. The lower path around the outer square is $\iota_{h \triangleright (g \triangleright N)} \circ \phi_h \circ \phi_g$. Applying this to $f : M \to a \triangleright M$ gives:

$$
\begin{aligned}
&\big(\iota_{h \triangleright (g \triangleright N)} \circ \phi_h \circ \phi_g\big)(f) \\
&= \iota_{h \triangleright (g \triangleright N)} \Big(\rho^1(h, gag^{-1}h^{-1})_{h \triangleright (g \triangleright M)}\Big)
\end{aligned}
$$

$\square$

Here is a repost of the classical character table for D_8:

Rep $\downarrow$ $\diagdown$ $[g] \rightarrow$	$\{1\}$	$\{a^2\}$	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
$\mathbb{1}$	1	1	1	1	1
I	1	1	-1	-1	1
J	1	1	-1	1	-1
IJ	1	1	1	-1	-1
V	2	-2	0	0	0

2Rep $\downarrow$ $\diagdown$ $[g] \rightarrow$	$\{1\}$	$\{a^2\}$	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
D_4	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1} \oplus IJ$	$\mathbb{1} \oplus J$	$\mathbb{1} \oplus I$
$\langle a \rangle$	$\mathbb{1} \oplus IJ$	$\mathbb{1} \oplus IJ$	$2(\mathbb{1} \oplus IJ)$	0	0
L_0	$\mathbb{1} \oplus J$	$\mathbb{1} \oplus J$	0	$2(\mathbb{1} \oplus J)$	0
L_0^ϕ	$\mathbb{1} \oplus J$	$I \oplus IJ$	0	$2V$	0
L_1	$\mathbb{1} \oplus I$	$\mathbb{1} \oplus I$	0	0	$2(\mathbb{1} \oplus I)$
L_1^ϕ	$\mathbb{1} \oplus I$	$J \oplus IJ$	0	0	$2V$
$\langle a^2 \rangle$	$\mathbb{1} \oplus I \oplus J \oplus IJ$	$\mathbb{1} \oplus I \oplus J \oplus IJ$	0	0	0
$\langle b \rangle$	$\mathbb{1} \oplus J \oplus V$	0	0	$\mathbb{1} \oplus J \oplus V$	0
$\langle ab \rangle$	$\mathbb{1} \oplus I \oplus V$	0	0	0	$\mathbb{1} \oplus I \oplus V$
$\{1\}$	$\mathbb{1} \oplus I \oplus J \oplus IJ \oplus 2V$	0	0	0	0

$\boxtimes$	D_4	$\langle a \rangle$	L_0	L_0^ϕ	L_1	L_1^ϕ	$\langle a^2 \rangle$	$\langle b \rangle$	$\langle ab \rangle$	$\{1\}$
D_4	D_4	$\langle a \rangle$	L_0	L_0^ϕ	L_1	L_1^ϕ	$\langle a^2 \rangle$	$\langle b \rangle$	$\langle ab \rangle$	$\{1\}$
$\langle a \rangle$	$\langle a \rangle$	$2\langle a \rangle$	$\langle a^2 \rangle$	$\langle a^2 \rangle$	$\langle a^2 \rangle$	$\langle a^2 \rangle$	$2\langle a^2 \rangle$	$\{1\}$	$\{1\}$	$2\{1\}$
L_0	L_0	$\langle a^2 \rangle$	$2L_0$	$2L_0^\phi$	$\langle a^2 \rangle$	$\langle a^2 \rangle$	$2\langle a^2 \rangle$	$2\langle b \rangle$	$\{1\}$	$2\{1\}$
L_0^ϕ	L_0^ϕ	$\langle a^2 \rangle$	$2L_0^\phi$	$2L_0$	$\langle a^2 \rangle$	$\langle a^2 \rangle$	$2\langle a^2 \rangle$	$2\langle b \rangle$	$\{1\}$	$2\{1\}$
L_1	L_1	$\langle a^2 \rangle$	$\langle a^2 \rangle$	$\langle a^2 \rangle$	$2L_1$	$2L_1^\phi$	$2\langle a^2 \rangle$	$\{1\}$	$2\langle ab \rangle$	$2\{1\}$
L_1^ϕ	L_1^ϕ	$\langle a^2 \rangle$	$\langle a^2 \rangle$	$\langle a^2 \rangle$	$2L_1^\phi$	$2L_1$	$2\langle a^2 \rangle$	$\{1\}$	$2\langle ab \rangle$	$2\{1\}$
$\langle a^2 \rangle$	$\langle a^2 \rangle$	$2\langle a^2 \rangle$	$2\langle a^2 \rangle$	$2\langle a^2 \rangle$	$2\langle a^2 \rangle$	$2\langle a^2 \rangle$	$4\langle a^2 \rangle$	$2\{1\}$	$2\{1\}$	$4\{1\}$
$\langle b \rangle$	$\langle b \rangle$	$\{1\}$	$2\langle b \rangle$	$2\langle b \rangle$	$\{1\}$	$\{1\}$	$2\{1\}$	$2\langle b \rangle \oplus \{1\}$	$2\{1\}$	$4\{1\}$
$\langle ab \rangle$	$\langle ab \rangle$	$\{1\}$	$\{1\}$	$\{1\}$	$2\langle ab \rangle$	$2\langle ab \rangle$	$2\{1\}$	$2\{1\}$	$2\langle ab \rangle \oplus \{1\}$	$4\{1\}$
$\{1\}$	$\{1\}$	$2\{1\}$	$2\{1\}$	$2\{1\}$	$2\{1\}$	$2\{1\}$	$4\{1\}$	$4\{1\}$	$4\{1\}$	$8\{1\}$

References

[CVOZ14] Huixiang Chen, Fred Van Oystaeyen, and Yinhuo Zhang. The Green rings of Taft algebras. *Proc. Amer. Math. Soc.*, 142(3):765–775, 2014.

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[EO18] Pavel Etingof and Victor Ostrik. On semisimplification of tensor categories. 2018.

[Gre64] J. A. Green. A transfer theorem for modular representations. *J. Algebra*, 1:73–84, 1964.

[hes] David E Speyer (https://mathoverflow.net/users/297/david-e speyer). Simultaneous upper-triangularization for two nilpotent commuting matrix. MathOverflow. URL:https://mathoverflow.net/q/295771 (version: 2018-03-21).

[Lei15] Tom Leinster. The bijection between projective indecomposable and simple modules. *Bull. Belg. Math. Soc. Simon Stevin*, 22(5):725–735, 2015.

[Shi17] Kenichi Shimizu. The monoidal center and the character algebra. *J. Pure Appl. Algebra*, 221(9):2338–2371, 2017.

[Shi19] Kenichi Shimizu. Non-degeneracy conditions for braided finite tensor categories. *Adv. Math.*, 355:106778, 36, 2019.