Pointed Braided Fusion $\leftrightarrow$ Quadratic Groups

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May 17, 2020

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Introduction

Pointed Braided Fusion ↔ Quadratic Groups

Main Ideas

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- A QG $= (G, q)$, with $q : G \to \mathbb{C}^\times$ and subject to equations.

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Main Ideas

- There is a correspondance between pointed braided fusion categories and quadratic groups.
- A generic PBF is $\simeq (\mathrm{Vec}_G^{\omega}, c_{-,-})$.
- A QG $= (G, q)$, with $q : G \to \mathbb{C}^{\times}$ and subject to equations.
- A whole lot of cohomology around here.

Pointed Fusion

A reminder of definitions:

Definition

A fusion category is a semisimple tensor category, with finitely many simple objects.

Definition

A tensor category is pointed when every simple object is invertible, *i.e.* pointed means: X is simple $\iff X^* \otimes X \xrightarrow{\cong} 1 \xrightarrow{\cong} X \otimes X^*$

$G(\mathcal{C})$

Pointed Fusion

- For any rigid monoidal category $\mathcal{C}$, the invertible objects 'behave like' a group.
- Modulo isomorphisms, they form an actual group, denoted $G(\mathcal{C})$.
- Pointed implies that all the simples represent elements in this group
- Fusion implies that all objects are just direct sums of these simples.
- Conclude that PFCs $\mathcal{C}$ 'look like' $\mathrm{Vec}_{G(\mathcal{C})}$

Vec_G

Pointed Fusion

- Semisimple category generated by simples for each $g \in G$, and where

$$\mathrm{Hom}(g, h) \cong \begin{cases} \mathbb{C} & \text{if } g = h \\ 0 & \text{otherwise} \end{cases}$$

- The tensor product is just $g \otimes h = gh$.
- All unitors and associators are trivial.

Skeletal vs Strict

Pointed Fusion

- Skeletal categories have only one object in each iso class.
- Strict categories have unitors and associators equal to identities.

Theorem (Mac Lane, *c.f.* EGNO18, Rmks 2.8.6-7)

Every category is equivalent to a skeletal category, and every monoidal category is equivalent to a strict monoidal category, ***BUT***

Skeletal vs Strict

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Theorem (Mac Lane, *c.f.* EGNO18, Rmks 2.8.6-7)

Every category is equivalent to a skeletal category, and every monoidal category is equivalent to a strict monoidal category, ***BUT*** *You cannot assume both at once!*

Why $\mathcal{C} \not\simeq \mathrm{Vec}_{G(\mathcal{C})}$

Pointed Fusion

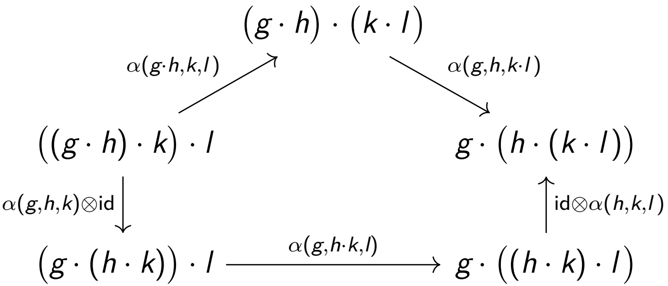
- The fusion rules match, but
- we never used the unitors or associators from $\mathcal{C}$. They might be nontrivial.
- It's possible for $\mathcal{C}$ to not be equivalent to a strict and skeletal category.
- $\mathrm{Vec}_{G(\mathcal{C})}$ *is strict and skeletal!*

Going Skeletal

Pointed Fusion

Our objects are g's. A skeleton of $\mathcal{C}$ is like $\mathrm{Vec}_{G(\mathcal{C})}$... except associators nontrivial.

$$\alpha(g, h, k) \in \mathrm{End}(ghk) \cong \mathbb{C}$$



Understanding α

Pointed Fusion

$$\alpha(gh, k, l)\alpha(g, h, kl) = \alpha(g, h, k)\alpha(g, hk, l)\alpha(h, k, l)$$

- This is a form of cocycle equation.
- We'll see that $[\alpha] \in H^3(G; \mathbb{C}^\times)$

Group Cohomology

Let G be a group and let A be an abelian group. $H^n(G;A)$ is an abelian group encoding certain 'higher dimensional information' about G.

1. $H^1(G;A) = \mathrm{Hom}(G,A)$
2. $H^2(G;A)$ records isomorphism classes of central extensions of G by A.
3. $H^3(G;A)$ classifies certain crossed modules and *associators*.
4. $H^n(G;A)$ for higher n is harder to pin down.

Group Cohomology

Definition

The group of n-cochains of G with coefficients in A is denoted $C^n(G;A)$ and consists of all functions from $G^n \to A$.

- There is a map $\delta : C^n(G;A) \to C^{n+1}(G;A)$ defined by

$$(\delta\phi)(g_1,\cdots,g_{n+1}) = \phi(g_2,\cdots,g_n) + \sum_{i=1}^{n}(-1)^i\phi(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}) + (-1)^{n+1}\phi(g_1,\ldots,g_n)$$

Group Cohomology

- These form a cochain complex:

$$0 \longrightarrow C^0(G;A) \xrightarrow{\delta} C^1(G;A) \xrightarrow{\delta} C^2(G;A) \xrightarrow{\delta} C^3(G;A) \xrightarrow{\delta} \cdots$$

- $H^n(G;A) := \dfrac{\ker\left(\delta : C^n \to C^{n+1}\right)}{\operatorname{im}\left(\delta : C^{n-1} \to C^n\right)}$
- For the topological, $H^n(G;A) \cong H^n(\mathbf{B}G;A)$, $\mathbf{B}G$ the classifying space of G.

Group Cohomology

with trivial coefficients

$(G; A)$	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7
$(\mathbb{Z}\,;\,\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0	0
$(\mathbb{Z}\,;\,\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	0	0	0
$(\mathbb{Z}\,;\,\mathbb{C}^\times)$	$\mathbb{C}^\times$	$\mathbb{C}^\times$	0	0	0	0	0	0
$(\mathbb{Z}/p\,;\,\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0
$(\mathbb{Z}/p\,;\,\mathbb{Z}/2)$	$\mathbb{Z}/2$	0	0	0	0	0	0	0
$(\mathbb{Z}/p\,;\,\mathbb{C}^\times)$	$\mathbb{C}^\times$	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$
$(S_3\,;\,\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/6$	0	$\mathbb{Z}/2$	0
$(S_3\,;\,\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$(S_3\,;\,\mathbb{C}^\times)$	$\mathbb{C}^\times$	$\mathbb{Z}/2$	0	$\mathbb{Z}/6$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/6$

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Group Cohomology

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$(G; A)$	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7
$(\mathbb{Z}; \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0	0
$(\mathbb{Z}; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	0	0	0
$(\mathbb{Z}; \mathbb{C}^\times)$	$\mathbb{C}^\times$	$\mathbb{C}^\times$	0	0	0	0	0	0
$(\mathbb{Z}/p; \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0
$(\mathbb{Z}/p; \mathbb{Z}/2)$	$\mathbb{Z}/2$	0	0	0	0	0	0	0
$(\mathbb{Z}/p; \mathbb{C}^\times)$	$\mathbb{C}^\times$	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$	0	$\mathbb{Z}/p$
$(S_3; \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/6$	0	$\mathbb{Z}/2$	0
$(S_3; \mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$(S_3; \mathbb{C}^\times)$	$\mathbb{C}^\times$	$\mathbb{Z}/2$	0	$\mathbb{Z}/6$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/6$

$H^3(G; \mathbb{C}^\times)$

Group Cohomology

$\mathbb{C}^\times$ is usually written multiplicatviely, so the 3-cocyle equation

$$0 = (\delta\alpha)(g,h,k,l) = \alpha(h,k,l) - \alpha(gh,k,l) + \alpha(g,hk,l) - \alpha(g,h,kl) + \alpha(g,h,k) ,$$

would be written as

$$1 = (\delta\alpha)(g,h,k,l) = \alpha(h,k,l) \cdot \alpha(gh,k,l)^{-1} \cdot \alpha(g,hk,l) \cdot \alpha(g,h,kl)^{-1} \cdot \alpha(g,h,k)$$

$$\Longleftrightarrow$$

$$\alpha(gh,k,l)\alpha(g,h,kl) = \alpha(g,h,k)\alpha(g,hk,l)\alpha(h,k,l)$$

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$\mathcal{C} \simeq \text{Vec}_G^\omega$

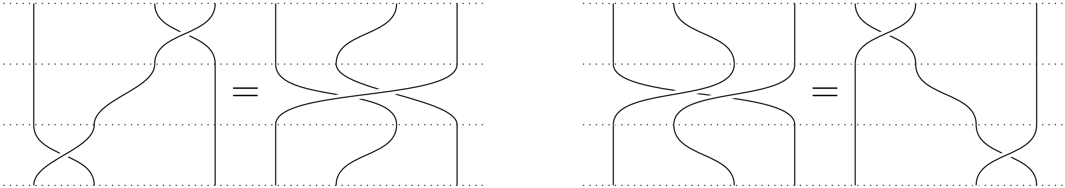
Group Cohomology

- Take $G = G(\mathcal{C})$.
- Take $\omega = [\alpha] \in H^3(G; \mathbb{C}^\times)$.
- This is a skeletalization that incorporates the monoidal structure correctly.
- Up to monoidal equivalence, *all pointed fusion categories are of this form!*

What about a Braiding on Vec_G^ω?

Group Cohomology

In order to encode a braiding, we need the associator α, as well as the braiding c, and we need them to satisfy the pentagon and the two hexagon relations.



Abelian Cocycles

Group Cohomology

Pentagon	$\iff$	$\alpha(gh,k,l)\alpha(g,h,kl) = \alpha(g,h,k)\alpha(g,hk,l)\alpha(h,k,l)$
Hexagon 1	$\iff$	$\alpha(h,k,g)c(g,hk)\alpha(g,h,k) = c(g,k)\alpha(h,g,k)c(g,h)$
Hexagon 2	$\iff$	$\alpha(k,g,h)^{-1}c(gh,k)\alpha(g,h,k)^{-1} = c(g,k)\alpha(g,k,h)^{-1}c(h,k)$

A solution (α, c) to these equations is called an abelian cocyle $\in Z^3_{ab}(G;\mathbb{C}^\times)$.

Ex 8.4.4

Group Cohomology

Suppose $\alpha(g,h,k)=1$. Show that c must be a bicharacter.

Pentagon	$\iff$	$\alpha(gh,k,l)\alpha(g,h,kl)=\alpha(g,h,k)\alpha(g,hk,l)\alpha(h,k,l)$
Hexagon 1	$\iff$	$\alpha(h,k,g)c(g,hk)\alpha(g,h,k)=c(g,k)\alpha(h,g,k)c(g,h)$
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Group Cohomology

Suppose $\alpha(g, h, k) = 1$. Show that c must be a bicharacter.

$$\boxed{\text{Pentagon}} \iff \alpha(gh, k, l)\alpha(g, h, kl) = \alpha(g, h, k)\alpha(g, hk, l)\alpha(h, k, l)$$

$$\boxed{\text{Hexagon 1}} \iff \alpha(h, k, g)c(g, hk)\alpha(g, h, k) = c(g, k)\alpha(h, g, k)c(g, h)$$

$$\boxed{\text{Hexagon 2}} \iff \alpha(k, g, h)^{-1}c(gh, k)\alpha(g, h, k)^{-1} = c(g, k)\alpha(g, k, h)^{-1}c(h, k)$$

Ex 8.4.4

Group Cohomology

Suppose $\alpha(g, h, k) = 1$. Show that c must be a bicharacter.

$$\boxed{\text{Pentagon}} \iff 1 = 1$$

$$\boxed{\text{Hexagon 1}} \iff c(g, hk) = c(g, k)c(g, h)$$

$$\boxed{\text{Hexagon 2}} \iff c(gh, k) = c(g, k)c(h, k)$$

Abelian Coboundaries

Group Cohomology

Abelian cochains fit into a cochain complex too!

- For $J : G \times G \to \mathbb{C}^\times$,

$$(\delta J)(g,h,k) = \left(J(h,k)J(gh,k)^{-1}J(g,hk)J(g,h)^{-1} \ , \ J(h,g)J(g,h)^{-1} \right)$$

- Such pairs of functions are called abelian coboundaries $\in B^3_{ab}(G;\mathbb{C}^\times)$.

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- **Boredom Check:** try to verify that $B^3_{ab}(G;\mathbb{C}^\times) \leq Z^3_{ab}(G;\mathbb{C}^\times)$.

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- **Boredom Check:** try to verify that $B^3_{ab}(G;\mathbb{C}^\times) \leq Z^3_{ab}(G;\mathbb{C}^\times)$.
- What diagrams do these formulas correspond to?

Braided Tensor Functors → Abelian Coboundaries

Group Cohomology

A braided tensor functor $(F, J) : (\mathrm{Vec}_G^{[\alpha]}, c) \to (\mathrm{Vec}_K^{[\beta]}, d)$ is a group homomorphism $F : G \to K$ and a natural isomorphism $J : F(g)F(h) \to F(gh)$ subject to the conditions below:

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Associator compatibility:

$$\begin{array}{ccc} (FgFh)Fk & \xrightarrow[F^*(\beta)(g,h,k)]{} & Fg(FhFk) \\ {\scriptstyle J(g,h)^{-1}\otimes \mathrm{id}}\uparrow & & \downarrow{\scriptstyle \mathrm{id}\otimes J(h,k)} \\ F(gh)Fk & & FgF(hk) \\ {\scriptstyle J(gh,k)^{-1}}\uparrow & & \downarrow{\scriptstyle J(g,hk)} \\ F((gh)k) & \xrightarrow{F\alpha(g,h,k)} & F(g(hk)) \end{array}$$

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F(gh)Fk & & FgF(hk) \\
{\scriptstyle J(gh,k)^{-1}}\uparrow & & \downarrow{\scriptstyle J(g,hk)} \\
F((gh)k) & \xrightarrow{F\alpha(g,h,k)} & F(g(hk))
\end{array}
$$

Braiding compatibility:

$$
\begin{array}{ccc}
FgFh & \xrightarrow{F^*(d)(g,h)} & FhFg \\
{\scriptstyle J(g,h)^{-1}}\uparrow & & \downarrow{\scriptstyle J(h,k)} \\
F(gh) & \xrightarrow{Fc(g,h)} & F(hg)
\end{array}
$$

Braided Tensor Functors → Abelian Coboundaries

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Associator $\Rightarrow$ $\alpha(g, h, k) = F^*(\beta)(g, h, k) \cdot J(h, k)J(gh, k)^{-1}J(g, hk)J(g, h)^{-1}$

Braiding $\Rightarrow$ $c(g, h) = F^*(d)(g, h) \cdot J(h, g)J(g, h)^{-1}$

$$\Longleftrightarrow$$

$$(\alpha, c) = F^*(\beta, d) \cdot \delta J$$

Conclusion: F pulls back the target cocycle to one that is cohomologous to the domain cocycle, as whitnessed by the tensorator J.

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Abelian Cohomology

Group Cohomology

Definition

The third abelian cohomology of G is

$$H^3_{ab}(G; \mathbb{C}^\times) := \frac{Z^3_{ab}(G; \mathbb{C}^\times)}{B^3_{ab}(G; \mathbb{C}^\times)}$$

Abelian Cohomology

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$$H^3_{ab}(G;\mathbb{C}^\times) := \frac{Z^3_{ab}(G;\mathbb{C}^\times)}{B^3_{ab}(G;\mathbb{C}^\times)}$$

We've established that

$$\left\{ [PBFC]^\times_\simeq \right\} \leftrightarrow \left\{ (G,\omega) \,\middle|\, \omega \in H^3_{ab}(G;\mathbb{C}^\times)/Aut(G) \right\}$$

Abelian Cocycles → Quadratic Functions

Group Cohomology

- Set $q(g) := c(g, g)$.
- Ab-Cocycle equations imply that

$$[\dagger] \quad q(g^{-1}) = q(g), \text{ and } \quad q(ghk)q(g)q(h)q(k) = q(gh)q(hk)q(kg)$$

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- This means $q : G \to \mathbb{C}^\times$ is *quadratic*.

Quadratic Groups

Definition

A quadratic group is a pair (G, q) with G an abelian group and $q : G \to \mathbb{C}^\times$ where for all $g, h, k \in G$,

$$q(g) = q(g^{-1}), \quad \text{and} \quad b(g,h) := \frac{q(gh)}{q(g)q(h)} \text{ is bimultiplicative, i.e.}$$

$$b(gh, k) = b(g,k)b(h,k) \text{ and } b(g, hk) = b(g,h)b(g,k).$$

A function q satisfying these properties said to be quadratic, and the function b is called the associated bicharacter.

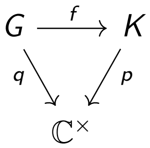
Quadratic Groups

Definition

The set of all quadratic functions $q : G \to \mathbb{C}^\times$ on a given group G, is denoted by $\mathcal{Q}uad(G)$ and inherits a group structure from the product in $\mathbb{C}^\times$.

Definition

A morphism of quadratic groups $f\colon (G, q) \to (K, p)$ is a group homomorphism such that $p \circ f = q$.



Break

Quadratic Groups

1. Show that $\alpha(g,h,k) = (-1)^{ghk}$ is a nontrivial element in $H^3(\mathbb{Z}/2;\mathbb{C}^\times)$.
2. Show that an monoidal isomorphism $\eta : (F_1, J_1) \to (F_2, J_2)$ of two braided functors $(F_1, J_1), (F_2, J_2) : (\mathrm{Vec}_G^{[\alpha]}, c) \to (\mathrm{Vec}_K^{[\beta]}, d)$ determines a 1-cochain $\lambda : G \to \mathbb{C}^\times$ such that $J_1 \cdot \delta\lambda = J_2$.
3. Show that the quadratic equations (†) are equivalent to (G, q) being a quadratic group.
4. Use either formulation to prove that $\mathcal{Q}uad(\mathbb{Z}) \cong \mathbb{C}^\times$.
5. Show that $\left(\mathbb{Z}/2, n \mapsto i^{n^2}\right)$ is a quadratic group.

A Surprising Isomorphism

Quadratic Groups

In the 1950s, Eilenberg and Mac Lane introduced H^*_{ab} and proved:

Theorem (EM50)

The assignment $[(\alpha, c)] \mapsto c \circ \delta$ *is an isomorphism:*

$$H^3_{ab}\left(G; \mathbb{C}^\times\right) \cong H^4\left(K(G,2); \mathbb{C}^\times\right) \cong \mathcal{Q}uad(G)$$

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Note: $H^4(K(\mathbb{Z},2); \mathbb{C}^\times) \cong H^4(\mathbb{C}P^\infty; \mathbb{C}^\times) \cong \mathbb{C}^\times$

Upgrading to an Equivalence

Proving the Equivalence

- PBFCs form a category with isomorphism classes of braided tensor functors as morphisms.
- Quadratic groups and morphisms form a category.

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Proving the Equivalence

- PBFCs form a category with isomorphism classes of braided tensor functors as morphisms.
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Theorem (JS)

The following assignment is an equivalence:

$$\begin{aligned} PBFCs &\longrightarrow \mathcal{Q}uad \\ (\mathcal{C}, c) &\longmapsto \big(G(\mathcal{C}), c \circ \Delta\big) \\ \Big((F, J) : \mathcal{C} \to \mathcal{D}\Big) &\longmapsto \Big(F_* : G(\mathcal{C}) \to G(\mathcal{D})\Big) \end{aligned}$$

Sketching the Proof

Proving the Equivalence

$$\left\{PBFCs\right\} \longleftrightarrow \mathcal{Q}uad$$

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Proving the Equivalence

$$\left\{PBFCs\right\} \leftrightarrow \left\{H^3_{ab}\right\} \leftrightarrow \mathcal{Q}uad$$

1. Essentially surjective
2. Full
3. Faithful

Sketching the Proof

Proving the Equivalence

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1. Essentially surjective ✓
2. Full
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Sketch Cont'd: Fullness

Proving the Equivalence

$$\left\{PBFCs\right\} \leftrightarrow \left\{H^3_{ab}\right\} \leftrightarrow Quad$$

Sketch Cont'd: Fullness

Proving the Equivalence

$$\left\{PBFCs\right\} \leftrightarrow \left\{H^3_{ab}\right\} \leftrightarrow \mathcal{Q}uad$$

1. Take a morphism of quadratic groups $F : (G, q) \to (K, p)$. $q \rightsquigarrow [(\alpha, c)]$ and $p \rightsquigarrow [(\beta, d)]$

.

Sketch Cont'd: Fullness

Proving the Equivalence

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1. Take a morphism of quadratic groups $F : (G, q) \to (K, p)$. $q \rightsquigarrow [(\alpha, c)]$ and $p \rightsquigarrow [(\beta, d)]$
2. We find $F^*[(\beta, d)] = [(\alpha, c)]$, so there is some 2-cochain J, w/ $F^*(\beta, d) \cdot \delta J = (\alpha, c)$

.

Sketch Cont'd: Fullness

Proving the Equivalence

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2. We find $F^*[(\beta, d)] = [(\alpha, c)]$, so there is some 2-cochain J, w/ $F^*(\beta, d) \cdot \delta J = (\alpha, c)$
3. Use F and J to define $(F, J) : (\mathrm{Vec}_G^{[\alpha]}, c) \to (\mathrm{Vec}_K^{[\beta]}, d)$.

Sketch Cont'd: Faithfulness

Proving the Equivalence

$$\left\{ PBFCs \right\} \leftrightarrow \left\{ H^3_{ab} \right\} \leftrightarrow \mathcal{Q}uad$$

Sketch Cont'd: Faithfulness

Proving the Equivalence

$$\left\{PBFCs\right\} \leftrightarrow \left\{H^3_{ab}\right\} \leftrightarrow \mathcal{Q}uad$$

1. Pick two braided tensor functors $(F_1, J_1), (F_2, J_2) : (\mathrm{Vec}_G^{[\alpha]}, c) \to (\mathrm{Vec}_K^{[\beta]}, d)$

Sketch Cont'd: Faithfulness

Proving the Equivalence

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2. Suppose they determine the same morphism of quadratic groups. In particular $F_1 = F_2 : G \to K$.

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Sketch Cont'd: Faithfulness

Proving the Equivalence

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4. $\implies J = \delta\lambda$ which means that λ can be used to define an isomorphism of the functors.

Conclusion & References

- Quadratic groups provide a *truncation* of the 2-category of PBFCs.
- If we only care about functors up to isomorphism, we might as well work with $\mathcal{Q}uad$.
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