Understanding Mod-R

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Abstract

This note is about the category Mod-R, where R is a ring. The goal is not to understand how the structure of the ring effects the internal structure of the category, but rather to understand the important role that categories of this form play within the 2-category of abelian categories.

Contents

1 Preliminaries

These notes focus on abelian and $\mathbb{K}$-linear categories where $\mathbb{K}$ is a commutative ring. A modern review of the important definitions can be found in chapter 1 of [2]. We will occasionally discuss nonabelian categories, and will strive to be transparent about this when it happens.

We begin by singling out three properties that an object $X \in \mathcal{C}$ may have.

Definition X is said to be a separator if $\mathcal{C}(X, -)$ is faithful.

Example 1.1 The following are separators for their respective categories:

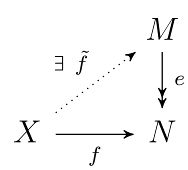
- $\mathbb{Z}$ in the category Ab of abelian groups.
- R in the category Mod-R.

- $\mathbb{K}G$ in the category $\mathsf{Rep}_{\mathbb{K}}G$ of representations of a group G.
- $\mathcal{O}_X$ in the category $\mathsf{coh}(X)$ of coherent sheaves on a scheme X.
- In Set, which is nonabelian, any nonempty set is a separator.

Definition X is said to be projective if $\mathcal{C}(X,-)$ is exact. (Any functor of this form is automatically left-exact, so this property is equivalent to the functor being right-exact.)

Proposition 1.1 *The following are equivalent:*

1. *X is projective,*
2. *$\mathcal{C}(X,-)$ preserves finite colimits,*
3. *(Lifting) For all epimorphisms $e : M \to N$ and morphisms $f : X \to N$, there is a map $\tilde{f} : X \to M$ such that $e\tilde{f} = f$.*



4. *X has the property that all exact sequences of the form*

$$A \hookrightarrow B \stackrel{q}{\twoheadrightarrow} X$$

are necessarily split, i.e. *there is a map $s : X \to B$ such that $qs = id_X$, and hence $B \cong A \oplus X$.*

Example 1.2

a. If there is an adjunction $F : \mathsf{Set} \rightleftarrows \mathcal{C} : U$ where U preserves epimorphisms, then objects in the image of F ('free' objects) are always projective.

b. Initial objects are always projective in the lifting sense, whether the category is abelian or not.

c. Any direct summand of a projective module is projective.

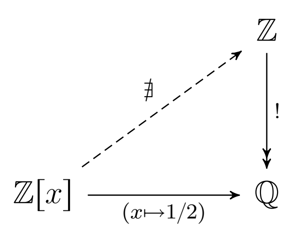
d. In RMod, the forgetful functor preserves epics, and all objects are quotients of free objects. This implies that an R-module is projective if and only if it is a direct summand of a free module.

e. For any locally small, additive category $\mathcal{C}$, the Yoneda embedding is defined by

$$\begin{aligned} Y : \mathcal{C} &\hookrightarrow \mathsf{Ab}^{\mathcal{C}^{op}} \\ X &\mapsto \mathcal{C}(-, X) \\ f &\mapsto f_*. \end{aligned}$$

The image $\mathcal{C}(-, X)$ of any object $X \in \mathcal{C}$ is projective as an object of the functor category $\mathsf{Ab}^{\mathcal{C}^{op}}$.

f. In the nonabelian category CRing of commutative rings, free objects are not necessarily projective in the lifting sense 3. There is a free-forgetful adjunction $F : \mathsf{Set} \rightleftarrows \mathsf{CRing} : U$. The inclusion of $\mathbb{Z}$ into $\mathbb{Q}$ is famously an epimorphism in CRing, but it is not surjective on underlying sets. In other words, U does not preserve epimorphisms. The ring $\mathbb{Z}[x]$ is free on the one element set, and the rule $x \mapsto 1/2$ determines a map of commutative rings $\mathbb{Z}[x] \to \mathbb{Q}$ that cannot be lifted in the following diagram:



Among other things, this shows that in example 1.2.a the requirement that U preserve epics is not superfluous.

g. Despite the previous example, $\mathbb{Z}$ is free on the empty set and *is* projective, since $\mathbb{Z}$ is initial in CRing and 1.2.b applies.

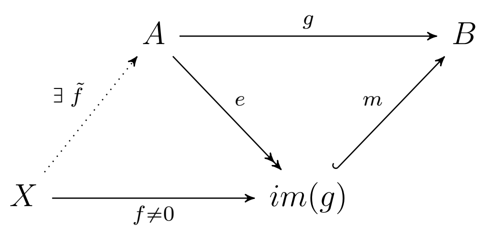
Proposition 1.2 *If $X \in \mathcal{C}$ is a separator, then $\mathcal{C}(X, A)$ is nonzero for every nonzero object $A \in \mathcal{C}$. If X is projective, then the converse is also true.*

Proof. Let X be a separator. Since A is nonzero, id_A is not the zero map. Since $\mathcal{C}(X, -)$ is faithful,

$$(\mathrm{id}_A)_* = \mathcal{C}(X, \mathrm{id}_A) = \mathrm{id}_{\mathcal{C}(X,A)}$$

is not the zero map either, and hence the abelian group $\mathcal{C}(X, A)$ is nonzero.

Now suppose X is projective and satisfies the desired property. Let $g : A \to B$ be a nonzero map. Since $\mathcal{C}$ is abelian, we can factor g into $g = me$, where $e : A \twoheadrightarrow \mathrm{im}(g)$ is the epimorphism to the image, and $m : \mathrm{im}(g) \hookrightarrow B$ is monic. The situation is modelled by the diagram below.



The fact that g is nonzero implies that the object $\mathrm{im}(g)$ is also nonzero. Using our hypothesis, we can conclude that $\mathcal{C}(X, \mathrm{im}(g))$ is nonzero. Thus, we can find some nonzero $f : X \to \mathrm{im}(g)$. Using projectivity, we can find a lift $\tilde{f} : X \to A$. Finally notice that

$$g_*(\tilde{f}) \;=\; g\tilde{f} \;=\; me\tilde{f} \;=\; mf \;\neq\; 0,$$

because $f \neq 0$ and m is monic. This tells us that g_* is nonzero, and hence $\mathcal{C}(X, -)$ is faithful, so X is a separator. □

Definition An object $X \in \mathcal{C}$ is called a (capable) generator if every object can be written as a quotient of a direct sum of (finitely many) copies of X.

Proposition 1.3 *Suppose $\mathcal{C}$ is $\mathbb{K}$-linear (with finitely generated hom-spaces) and has (finite) direct sums. In this setting, $X \in \mathcal{C}$ is a separator if and only if X is a (capable) generator.*

Proof. ($\Rightarrow$) Let $A \in \mathcal{C}$ be a generic object, and let $\{a_i\}_{i \in I}$ be a generating set for $\mathcal{C}(X, A)$. Define the map $e : \bigoplus_{i \in I} X \to A$ to be the coproduct $e = [a_i]_i$, and let $g : A \to B$ be a nonzero map. Since X is a separator, $g_* : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ is nonzero, and hence there is some $f : X \to A$ so that $gf \neq 0$. We can then write f as

$$f = \sum_{i \in I} k_i \cdot a_i,$$

and this implies that

$$\sum_{i \in I} k_i \cdot (g a_i) = g \left(\sum_{i \in I} k_i \cdot a_i \right)$$
$$= g f$$
$$\neq 0.$$

From this we can conclude that there is at least one $j \in I$ such that $(g a_j) \neq 0$, and this forces $ge \neq 0$. Since g was an arbitrary nonzero morphism, we conclude that e is an epimorphism, and that X is a generator.

($\Leftarrow$) Let $g : A \to B$ be a generic nonzero morphism, and take any epic $e : \bigoplus_{i \in I} X \twoheadrightarrow A$. Since $g \neq 0$, $ge \neq 0$ either. Since ge is a nonzero map out of a coproduct, it must be the case that there is some $j \in I$ such that

$$0 \neq (ge)\iota_j = g_*(e\iota_j)$$
$$\Longrightarrow$$
$$g_* \neq 0.$$

We have thus shown that $\mathcal{C}(X, -)$ is faithul, so X is a separator.

Finally, when $\mathcal{C}$ is enriched over finitely generated $\mathbb{K}$-modules, the indexing set I can be taken to be finite, and this proves the statement about capable generators. □

Note: Proposition 1.3 is fairly general and does not require $\mathcal{C}$ to be abelian. This connection between generators and separators is so often valid that many authors use the terms interchangeably.

Definition X is said to be compact if $\mathcal{C}(X, -)$ preserves filtered colimits.

Note: X compact implies in particular that

$$\mathcal{C}\Big(X\,,\, \bigoplus_{i \in I} Y_i\Big) \cong \bigoplus_{i \in I} \mathcal{C}\,(X\,,\, Y_i)$$

Proposition 1.4 *A finite colimit of compact objects is compact.*

Proof. Let $X = \mathsf{colim}_J\, X_j$ be a finite colimit of compact objects, and let I

be filtered. Observe that

$$\begin{aligned}
\mathcal{C}\left(X, \operatorname*{colim}_I Y_i\right) &= \mathcal{C}\left(\operatorname*{colim}_J X_j, \operatorname*{colim}_I Y_i\right) \\
&\cong \lim_J \mathcal{C}\left(X_j, \operatorname*{colim}_I Y_i\right) \\
&\cong \lim_J \operatorname*{colim}_I \mathcal{C}\Big(X_j, Y_i\Big) \\
&\overset{*}{\cong} \operatorname*{colim}_I \lim_J \mathcal{C}\Big(X_j, Y_i\Big) \\
&\cong \operatorname*{colim}_I \mathcal{C}\left(\operatorname*{colim}_J X_j, Y_i\right) \\
&= \operatorname*{colim}_I \mathcal{C}\Big(X, Y_i\Big).
\end{aligned}$$

The isomorphsim $(*)$ comes from the classical fact that finite limits commute with filtered colimits in $\mathbb{K}$-Mod (see for example Thm 2.6.15 from [4]). □

2 The Recognition Principles

In this section we outline multiple recognition principles that determine various ways in which an abelian category can be related to the category Mod-R for some ring R.

Theorem 2.1 *Let $\mathcal{C}$ be a $\mathbb{K}$-linear, abelian category with (arbitrary) direct sums. If $X \in \mathcal{C}$ is a compact, projective generator, then $\mathcal{C} \simeq$ Mod-R for the ring $R := \mathsf{End}(X)$. If $\mathcal{C}$ is only assumed to have finite direct sums, and X is capable instead of compact, then $\mathcal{C} \simeq$ mod-R, the category of finitely generated right R-modules.*

Proof. Firstly, notice that precomposition endows $\mathcal{C}(X, A)$ with the structure of a right R-module, and the functor $\mathcal{C}(X, -)$ actually factors through Mod-R. Once this is understood, the proof follows the following simple recipe:

a) Generator $\Longrightarrow$ Faithful

b) Generator + Compact $\Longrightarrow$ Full

c) Compact + Full + Projective $\Longrightarrow$ Essentially Surjective

That these three properties are necessary and sufficient for $\mathcal{C}(X,-) : \mathcal{C} \to \mathsf{Mod}\text{-}R$ to be an equivalence is a classical result of basic category theory (see *e.g.* [3]).

Step (a) is taken care of using 1.3, so let us proceed to step (b). Let

$$L : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$$

be a morphism of right R-modules. Using the generator condition, find an exact sequence of the form

$$X^{\oplus m} \xrightarrow{\ t\ } X^{\oplus n} \xrightarrow{\ e\ } A,$$

where m, n are possibly infinite cardinals. Next, apply L to each of the components of e, to obtain a map $[L(e\iota_i)]_i : X^{\oplus n} \to B$. We claim that $[L(e\iota_i)]_i t = 0$, and hence this new map factors through the cokernel of t (*a.k.a.* the object A). To prove our claim, it will suffice to show that for all j, the component $[L(e\iota_i)]_i t\iota_j$ is equal to zero.

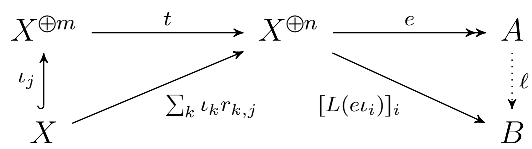
Using compactness, the map $t\iota_j : X \to X^{\oplus n}$ can be written as

$$t\iota_j \;=\; \sum_k \iota_k r_{k,j},$$

for some collection $r_{k,j} \in R$. Having this expression for $t\iota_j$ allows us to calculate:

$$\begin{aligned}
[L(e\iota_i)]_i t\iota_j &= [L(e\iota_i)]_i \left(\sum_k \iota_k r_{k,j} \right) \\
&= \sum_k L(e\iota_k) r_{k,j} \\
&\overset{*}{=} L\left(\sum_k e\iota_k r_{k,j} \right) \\
&= L\left(et\iota_j \right) \\
&= L(0) \\
&= 0,
\end{aligned}$$

where at $(*)$ we used the fact that L is a module homomorphism. Thus we conclude that there is some map $\ell : A \to B$ that makes the diagram commute.



Finally, let $f : X \to A$ be a generic map. A quick glance back to the proof of 1.3 will show that we can assume that e is the coproduct of generating elements for $\mathcal{C}(X, A)$. Thus f factors through $X^{\oplus n}$ as $f = e\tilde{f}$, where

$$f = \sum_k c_k \cdot f_k \quad \& \quad \tilde{f} = \sum_k c_k \cdot \iota_k$$

This allows us to perform the following calculation

$$\begin{aligned}
\ell_*(f) &= \ell f \\
&= \ell e \tilde{f} \\
&= [L(e\iota_i)]_i \tilde{f} \\
&= [L(e\iota_i)]_i \left(\sum_k c_k \cdot \iota_k \right) \\
&= \sum_k c_k \cdot L(e\iota_k) \\
&= L\left(\sum_k c_k \cdot e\iota_k \right) \\
&= L\left(e\tilde{f} \right) \\
&= L(f) \\
&\implies \ell_* = L.
\end{aligned}$$

This completes the proof of step (b).

For step (c), let M be an arbitrary right R-module. Consider an exact sequence of the form

$$R^{\oplus m} \xrightarrow{s_*} R^{\oplus n} \xrightarrow{q} M.$$

Since X is compact, $R^{\oplus k} \cong \mathcal{C}(X, X^{\oplus k})$ is in the image of our functor. By part (b), we are justified in writing s_* in the exact sequence above, as there must be some $s : X^{\oplus m} \to X^{\oplus n}$ whose image under $\mathcal{C}(X, -)$ is the map above. Define C to be the cokernel of s. Since X is projective, $\mathcal{C}(X, C) \cong M$, and our functor is essentially surjective.

Thus we have proven that $\mathcal{C}(X,-) : \mathcal{C} \to \mathsf{Mod}\text{-}R$ is an equivalence. To understand the alternative version, note that the above proof goes through *mutatis mutandis* since the only direct sums involved are finite. □

After recognizing that yours are categories of modules, the following theorem helps recognize when a functor is just a tensor product.

Theorem 2.2 (Eilenberg-Watts) *Given a functor $F : \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S$, $F(R)$ has a canonical R-S-bimodule structure. If F is right exact and preserves small direct sums, then $F \cong -\otimes_R F(R)$.*

Before going into the proof of 2.2, we would like to point out an important corollary:

Corollary 2.2.1 *Any right exact functor that preserves small direct sums $F : \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S$ between module categories has a right adjoint, which is given by $\mathsf{Mod}\text{-}S(F(R),-)$.*

This result is important because it gives an explicit formula for the right adjoint. Typically theorems such as the GAFTA, or SAFTA that guarantee the existence of an adjoint require you to exhibit a '*solution set*' in order to be constructive. The existence of solution sets is often verifiable without being computationally viable, and having the hom-set formula of 2.2.1 is what makes categories of modules so valuable.

Proof of Theorem 2.2. Observe that for any $r \in R$, left multiplication by r defines a map $\lambda_r : R \to R$ that is a right R-module homomorphism. This allows us to define a left R-module structure on the abelian group $F(R)$ by the formula $r.a := F(\lambda_r)(a)$. The right S-module structure is by assumption, and thus $F(R)$ can be given the structure of an R-S-bimodule.

Similarly to the above, for any $x \in M \in \mathsf{Mod}\text{-}R$, left multiplication by x:

$$\begin{aligned} \lambda_x : R &\to M \\ r &\mapsto x.r \end{aligned}$$

defines a right R-module homomorphism. This allows us to define the following map:

$$\begin{aligned} \phi_M : M \otimes_R F(R) &\to F(M) \\ x \otimes_R a &\mapsto F(\lambda_x)(a). \end{aligned}$$

The reader should verify that this is well-defined, *i.e.* that ϕ_M is R-balanced. This definition doesn't fundamentally use any structure of M, and so we can similarly define ϕ_N for any $N \in \mathsf{Mod}\text{-}R$. Furthermore, this family $\phi := \{\phi_N\}$ actually defines a natural morphism $\phi : - \otimes_R F(R) \to F$. To see this, let $f : M \to N$ and check:

$$\begin{aligned}
\big(F(f) \circ \phi_M\big)(x \otimes_R a) &= F(f)\big(F(\lambda_x)(a)\big) \\
&= \big(F(f) \circ F(\lambda_x)\big)(a) \\
&= \big(F(f \circ \lambda_x)\big)(a) \\
&\overset{\star}{=} \big(F(\lambda_{f(x)})\big)(a) \\
&= \phi_N\big(f(x) \otimes_R a\big) \\
&= \big(\phi_N \circ (f \otimes_R \mathrm{id})\big)(x \otimes_R a) \\
&\Longrightarrow \\
F(f) \circ \phi_M &= \phi_N \circ (f \otimes_R \mathrm{id}),
\end{aligned}$$

where $\star$ follows from the fact that f is a module homomorphism.

We wish to show that ϕ is a natural isomorphism of functors. Let us first consider the special case of free R modules. Observe that since F preserves direct sums, when applied to free modules, ϕ is realized by the following composition of isomorphisms:

$$\begin{aligned}
R^{\oplus n} \otimes_R F(R) &= \left(\bigoplus_{i=1}^{n} R\right) \otimes_R F(R) \\
&\cong \bigoplus_{i=1}^{n} \big(R \otimes_R F(R)\big) \\
&\cong \bigoplus_{i=1}^{n} F(R) \\
&\cong F\big(R^{\oplus n}\big).
\end{aligned}$$

Now let us take the first two terms of a free resolution of M,

$$R^m \longrightarrow R^n \twoheadrightarrow M$$

and apply our two functors to to arrive at the following commutative diagram:

$$\begin{array}{ccccc}
R^m \otimes_R F(R) & \longrightarrow & R^n \otimes_R F(R) & \twoheadrightarrow & M \otimes_R F(R) \\
\wr\big\downarrow \phi_{R^m} & & \wr\big\downarrow \phi_{R^n} & & \big\downarrow \phi_M \\
F\big(R^m\big) & \longrightarrow & F\big(R^n\big) & \twoheadrightarrow & F\big(M\big)
\end{array}$$

Here the rows remain right exact because of our assumption on F. Since the left two terms in each row are free, the left two vertical arrows are isomorphisms by our analysis of the special case. Finally by a diagram chase, ϕ_M must also be an isomorphism. □

References

[1] James F. Davis and Paul Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

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[4] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.