Notes on the J-Homomorphism

Sean Sanford

Abstract

The goal of these notes is to exposit on the J-homomorphism, in a way that is as transparent as possible. Given any map $f : S^k \to O(n)$, the J-construction produces a new map $J(f) : S^{k+n} \to S^n$. This procedure is often presented via a verbal description which does not allow for many computations, if any. We present an explicit formula for the J-construction, and prove that it induces a well-defined homomorphism $J : \pi_k\big(O(n)\big) \to \pi_{k+n}(S^n)$.

1 Preliminaries

The reader is expected to be familiar with elementray constructions from point-set topology, the notion of homotopy, and the higher homotopy groups of a space. At some point, there will be a technical lemma that uses the long exact sequence of a pair, cellular homology and the Hurewicz Theorem. For those less familiar with these concepts, the lemma is geometrically believable, and you are encouraged to take it as a black box on first pass. We begin our discussion with a few constructions that we will need.

We will use the notation $I = [0, 1]$ for the unit interval. The variable name t will be consistently used for points in I. Given a space X,

Definition. The *cone on* X is the space

$$CX := (I \times X)/\{(0, x)\}.$$

There is a map

$$\begin{aligned} H : CX \times I &\longrightarrow CX \\ \big([t, x], t'\big) &\longmapsto [(t't), x] \end{aligned}$$

which exhibits the one point subspace $\{[0,x]\}$ as a deformation retract of CX. This shows that CX is contractible for every space X.

Example 1.1. The cone on the n-sphere can be identified with the unit disk

$$CS^n \cong D^{n+1},$$

and the cone on an n-simplex can be identified with an $n+1$-simplex

$$C\Delta^n \cong \Delta^{n+1}.$$

From the cone, many interesting spaces can be defined. Note that there is an inclusion $\iota : X \hookrightarrow CX$ defined by $x \mapsto [1,x]$, which allows us to identify X with the subset $[1,X] \subseteq CX$.

Definition. The *suspension* of a space X is defined to be

$$\Sigma X := CX/X,$$

and the suspension of a map $f : X \to Y$ is defined to be

$$\begin{aligned} \Sigma f : \Sigma X &\longrightarrow \Sigma Y \\ [t,x] &\longmapsto [t,f(x)]. \end{aligned}$$

Here and throughout these notes we use square brackets to denote the equivalence class of elements in a product. Whether this is in the cone or the suspension should be clear from context. As defined, this forms a functor

$$\Sigma : \mathcal{S} \to \mathcal{S}$$

from the category of spaces to itself.

Example 1.2. The suspension of an n-sphere is homeomorphic to the $n+1$-sphere

$$\Sigma S^n \cong S^{n+1}.$$

Now let X and Y be spaces. We can construct

Definition. The *Join* of X and Y is the space

$$X * Y := I \times X \times Y / \sim$$

where the quivalence relation $\sim$ is that

$$\forall x, x' \in X,\ y, y' \in Y \quad (0,x,y) \sim (0,x',y) \ \&\ (1,x,y) \sim (1,x,y').$$

In a quip, the join could be described as:

'At 0 the x's don't matter, and at 1 the y's don't matter.'

This construction can be defined in a number of different ways, but this is the most economical for our purposes. Typical coordinates on the join will be denoted by $[t; x, y] \in X * Y$.

For these notes we will be particularly interested in the following example:

Example 1.3. The map

$$\begin{aligned} S^k * S^{n-1} &\longrightarrow S^{n+k} \\ [t; x, y] &\longmapsto tx + \sqrt{1-t^2}y \end{aligned}$$

is a homeomorphism. In order to make sense of the addition, we are thinking of $S^m \subseteq \mathbb{R}^{m+1}$, for $m \in \{k, n-1, k+n\}$ and then identifying $\mathbb{R}^{k+1} \times \mathbb{R}^n \cong \mathbb{R}^{n+k+1}$. Continuity should be clear, and the inverse mapping is given by

$$(x, y) \longmapsto \begin{cases} \left[0 \; ; \; x \; , \; \frac{y}{\|y\|}\right] & \text{if } \; 0 = \|x\| \\ \left[\|x\| \; ; \; \frac{x}{\|x\|} \; , \; \frac{y}{\|y\|}\right] & \text{if } \; 0 < \|x\| < 1 \\ \left[1 \; ; \; \frac{x}{\|x\|} \; , \; y\right] & \text{if } \; \|x\| = 1. \end{cases}$$

Closer inspection of this example shows that

$$\begin{aligned} \{[t; x, y] \in S^k * S^{n-1} \mid 0 \leqslant t \leqslant \tfrac{1}{2}\} &\cong D^{k+1} \times S^{n-1} \\ \{[t; x, y] \in S^k * S^{n-1} \mid \tfrac{1}{2} \leqslant t \leqslant 1\} &\cong S^k \times D^n. \end{aligned}$$

Thus we have a decomposition of $k+n$-sphere as

$$S^{k+n} \cong D^{k+1} \times S^{n-1} \cup S^k \times D^n.$$

When thinking about the join, it is often helpful to think of it as containing $X \cong [1; X, Y]$ at one end and $Y \cong [0; X, Y]$ at the other. In between these two ends lies a subspace $\cong (0,1) \times X \times Y$. Collapsing each end separately gives rise to a quotient map

$$X * Y \xrightarrow{c} \big((X * Y)/X\big)/Y \;\cong\; \Sigma\big(X \times Y\big).$$

Definition. To any map $f : X \times Y \to Z$ we can associate the map

$$(\Sigma f) \circ c : X * Y \to \Sigma Z.$$

The assignment $f \mapsto (\Sigma f) \circ c$ is known as the Hopf construction, and was introduced by Heinz Hopf in [2].

Definition. Given a map $f : X \to \mathcal{C}(Y, Y)$, it's *adjoint* $\tilde{f} : X \times Y \to Y$ is defined by the formula

$$\tilde{f}(x, y) := f(x)(y).$$

Example 1.4. The action of the orthogonal group $O(n)$ on $\mathbb{R}^n$ preserves lengths, and so $O(n)$ acts on the unit sphere S^{n-1}. This action allows us to think of $O(n)$ as a subspace of the space $\mathcal{C}(S^{n-1}, S^{n-1})$ of continuous maps from S^{n-1} to itself. Thus any map $f : S^k \to O(n) \subseteq \mathcal{C}(S^{n-1}, S^{n-1})$ has an adjoint $\tilde{f}$ whose formula looks like

$$\tilde{f}(x, y) = f(x)y,$$

where we use juxtaposition to indicate a matrix acting on a vector.

2 The Definition of the J-Homomorphism

Now we come to the main idea: the J-construction. Here we give two definitions primarily to indicate that the famous J-homomorphism is really just the map on homotopy classes induced by the general J-construction, which is well-defined on a point-set level.

Definition. The *J-construction for/applied to* $f : S^k \to O(n)$ is the map $J(f) : S^{k+n} \to S^n$ determined by the Hopf construction applied to the adjoint $\tilde{f}$. Explicitly, the formula is

$$\begin{aligned} J : \mathcal{C}(S^k, O(n)) &\longrightarrow \mathcal{C}(S^{k+n}, S^n) \\ f &\longmapsto (\Sigma \tilde{f}) \circ c. \end{aligned}$$

This definition only makes sense in light of our previous observations that $S^{k+n} \cong S^k * S^{n-1}$ and $\Sigma S^{n-1} \cong S^n$. If we use join coordinates for the domain and suspension coordinates for the range, then the map $J(f)$ becomes more transparent:

$$J(f) : [t; x, y] \longmapsto [t, f(x)y].$$

Let us assign the base points

$$\begin{aligned}(1,0,\ldots,0) =: p \;&\in\; S^k \\ \mathrm{id} &\in O(n) \\ [1;p,y] \;&\in\; S^k * S^{n-1} \cong S^{k+n} \\ [1,y] \;&\in\; \Sigma S^{n-1} \cong S^n.\end{aligned}$$

With these choices in place, observe that

$$J(f)\big([1;p,y]\big) \;=\; [1, f(p)y] \;=\; [1,y].$$

This shows that $J(f) : S^{k+n} \to S^n$ is a pointed map whether $f : S^k \to O(n)$ was pointed or not.

Definition. Restricting the J-construction results in a function

$$J_0 : \mathcal{C}_0(S^k, O(n)) \longrightarrow \mathcal{C}_0(S^{k+n}, S^n)$$

taking pointed maps to pointed maps. This induces a map on homotopy classes which is known as the *J-Homomorphism*:

$$J : \pi_k\big(O(n)\big) \longrightarrow \pi_{k+n}\big(S^n\big).$$

(Note: We also denote this function by J, as is tradition, though in our notation it might more accurately be denoted $(J_0)_*$.)

At this point, we have put two carts before the horse. Let us backtrack briefly.

Proposition 1. *The J-homomorphism is well-defined.*

Proof. If we let $H : S^k \times I \to O(n)$ be a pointed homotopy from $f = H(-,0)$ to $g = H(-,1)$, then we can define

$$\begin{aligned}K : S^{k+n} &\longrightarrow S^n \\ \big([t;x,y],t'\big) &\longmapsto [t, H(x,t')y].\end{aligned}$$

The map K is evidently continuous and pointed, and satisfies $K(-,0) = J(f)$ and $K(-,1) = J(g)$. Thus J_0 takes homotopic maps to homotopic maps as claimed, so J is well-defined. $\square$

We should also show that this map respects the group structure. This proof is due to Whitehead, and is pretty slick (see [4], or [3] for his original paper). First, we will need the technical lemma that we threatened about in the abstract. Here is the set up:

Let C be a CW-complex given by the following construction: attach an $m-1$-cell e^{m-1} to a single 0-cell e^0, then attach three m-cells e_1^m, e_2^m and e_3^m all along e^{m-1}. Let x_i, $i \in \{1, 2, 3\}$ be orientations for the e_i^m, each having the property that

$$\partial x_i \;=\; s$$

is the same orientation for e^{m-1}. The subsets $S_{ij}^m = e_i^m \cup e_j^m \cup e^{m-1} \cup e^0$, $1 \leqslant i \leqslant j \leqslant 3$ are m-spheres, and each will be oriented as $x_i - x_j$.

Lemma 1. *Given any pointed map $f : C \to X$ the homotopy classes $[f|_{S_{ij}^m}] =: \alpha_{ij} \in \pi_m(X)$ satisfy the relation*

$$\alpha_{13} = \alpha_{12} + \alpha_{23}.$$

Proof. By functoriality of π_m, it will suffice to prove the case $f = \mathrm{id} : C \to C$. Since C is $m-1$-connected, the Hurewicz map $\rho : \pi_m(C) \to H_m(C)$ is an isomorphism. Since $H_m(C^{(m-1)}) = 0$, the long exact sequence in homology shows that the promotion map

$$j : H_m(C) \hookrightarrow H_m(C, C^{(m-1)})$$

is injective. This implies that $j \circ \rho : \pi_m \to H_m(C, C^{(m-1)})$ is injective, but then

$$j \circ \rho(\alpha_{ij}) = x_i - x_j.$$

Finally since

$$\begin{aligned} j \circ \rho(\alpha_{13}) &= (x_1 - x_3) \\ &= (x_1 - x_2) + (x_2 - x_3) \\ &= j \circ \rho(\alpha_{12}) + j \circ \rho(\alpha_{23}), \end{aligned}$$

injectivity of $j \circ \rho$ implies the desired relation. □

Proposition 2. *The J-homomorphism is indeed a homomorphism.*

Proof. We will reference the following subsets of D^{k+1}:

$$\begin{aligned} S^k_+ &:= \{x \in S^k \mid x_{k+1} \geqslant 0\} \quad\subseteq\quad D^{k+1}_+ := \{x \in D^{k+1} \mid x_{k+1} \geqslant 0\} \\ S^k_- &:= \{x \in S^k \mid x_{k+1} \leqslant 0\} \quad\subseteq\quad D^{k+1}_- := \{x \in D^{k+1} \mid x_{k+1} \leqslant 0\}. \end{aligned}$$

Luckily, since the $O(n)$ are topological groups, the fundemental groups of $O(n)$ for every $n \geqslant 1$ are abelian. This is convenient for us, because it means that we can treat all spheres using a single argument.

Let $[f], [g] \in \pi_k\big(O(n)\big)$. Since they are pointed maps, may assume that $f(x) = \mathrm{id} \in O(n)$ for all $x \in S^k_-$ and that $g(x) = \mathrm{id}$ for all $x \in S^k_+$. The map

$$h : x \mapsto \begin{cases} f(x) & \text{if } \; x \in S^k_+ \\ g(x) & \text{if } \; x \in S^k_- \end{cases}$$

is then a representative of the sum $[h] = [f] + [g] \in \pi_k\big(O(n)\big)$. Our decomposition of D^{k+1} into halves yields the following decomposition of S^{k+n}:

$$\begin{aligned} S^{k+n} &\cong D^{k+1} \times S^{n-1} \cup S^k \times D^n \\ &\cong \big(D^{k+1}_- \cup D^{k+1}_+\big) \times S^{n-1} \cup \big(S^k_- \cup S^k_+\big) \times D^n \\ &\cong D^{k+1}_- \times S^{n-1} \cup D^{k+1}_+ \times S^{n-1} \cup S^k_- \times D^n \cup S^k_+ \times D^n \\ &\cong \big(D^{k+1}_- \times S^{n-1} \cup S^k_- \times D^n\big) \cup \big(D^{k+1}_+ \times S^{n-1} \cup S^k_+ \times D^n\big) \\ &=: S^{k+n}_- \cup S^{k+n}_+. \end{aligned}$$

This fits our naming convention in that these two subspaces are each homeomorphic to $k+n$-disks and S^{k+n} is their union along their common boundary which is homeomorphic to S^{k+n-1}. To see the claim about the boundary for example, observe the following calculation.

$$\begin{aligned} S^{k+n}_- \cap S^{k+n}_+ &= \big(D^{k+1}_- \times S^{n-1} \cup S^k_- \times D^n\big) \cap \big(D^{k+1}_+ \times S^{n-1} \cup S^k_+ \times D^n\big) \\ &= \big(D^{k+1}_- \times S^{n-1} \cap D^{k+1}_+ \times S^{n-1}\big) \cup \big(S^k_- \times D^n \cap D^{k+1}_+ \times S^{n-1}\big) \\ &\qquad \cup \big(D^{k+1}_- \times S^{n-1} \cap S^k_+ \times D^n\big) \cup \big(S^k_- \times D^n \cap S^k_+ \times D^n\big) \\ &= \big(D^k \times S^{n-1}\big) \cup \big(S^{k-1} \times S^{n-1}\big) \\ &\qquad \cup \big(S^{k-1} \times S^{n-1}\big) \cup \big(S^{k-1} \times D^n\big) \\ &= D^k \times S^{n-1} \cup S^{k-1} \times D^n \\ &\cong S^{k+n-1}. \end{aligned}$$

Using this decomposition, we can check that

$$\begin{aligned}
J(h)\big([t;x,y]\big) &= J(f)\big([t;x,y]\big) && \forall\ [t;x,y] \in S_+^{k+n} \\
J(f)\big([t;x,y]\big) &= [t,y] && \forall\ [t;x,y] \in S_-^{k+n} \\
J(g)\big([t;x,y]\big) &= [t,y] && \forall\ [t;x,y] \in S_+^{k+n} \\
J(h)\big([t;x,y]\big) &= J(g)\big([t;x,y]\big) && \forall\ [t;x,y] \in S_-^{k+n}.
\end{aligned}$$

Using the complex C from the lemma with $m = k + n$, we can define a map $f : C \to S^n$ that acts by the above three maps on the three different $k + n$-cells of C. The lemma then tells us that $J(h)$ represents the class $[J(f)] + [J(g)]$. Thus we have

$$\begin{aligned}
J\big([f] + [g]\big) &= J\big([h]\big) \\
&:= \big[J(h)\big] \\
&= [J(f)] + [J(g)] \\
&=: J\big([f]\big) + J\big([g]\big).
\end{aligned}$$

□

3 Two Explicit Computations

Example 3.1. Let

$$\mathrm{id}_0 : \mathbb{Z}/2 \cong S^0 \to O(1) \cong \mathbb{Z}/2$$

be the map corresponding to the identity on $\mathbb{Z}/2 = \{\pm 1\}$. Then we have that

$$\begin{aligned}
J(\mathrm{id}_0) : S^1 &\to S^1 \\
[t;\varepsilon,\nu] &\mapsto [t,\varepsilon\nu].
\end{aligned}$$

Here the $\varepsilon, \nu \in \{\pm 1\}$ correspond to the signs of the x and y coordinates of the unit circle in $\mathbb{R}^2$, so they are essentially telling you what quadrant you are in. the S^0 coordinate in the range indicates only the sign of the y-coordinate. As you run around the circle one full loop, the product $\varepsilon\nu$ goes from positive to negative twice. In $\mathbb{R}/\mathbb{Z}$ coordinates this is the $\times 2$ map, and in $S^1 \subseteq \mathbb{C}^\times$ coordinates, this is the $z \mapsto z^2$ map.

Example 3.2. Let

$$\mathrm{id}_1 : S^1 \to SO(2) \cong S^1 \leqslant \mathbb{C}^\times$$

be the map corresponding to the identity on S^1. Then we have that

$$\begin{aligned} J(\mathrm{id}_1) : S^3 &\to S^2 \\ [t; z, w] &\mapsto [t, zw]. \end{aligned}$$

The preimage of the north pole is $\{[1; z, w]\} \cong S^1$ (since only z matters when $t = 1$). Similarly the preimage of the south pole is also $\cong S^1$. Now, choosing a generic $[t, v]$ gives preimages that look like

$$\{[t; z, vz^{-1}]\},$$

so again we get circular preimages, since there is only one free coordinate. This observation can be used to construct local trivializations:

$$\begin{aligned} J(\mathrm{id}_1)^{-1}(S^2_+) &\overset{\cong}{\longleftrightarrow} S^2_+ \times S^1 \\ [t; z, w] &\longmapsto \big([t, zw], z\big), \quad \& \end{aligned}$$

$$\begin{aligned} J(\mathrm{id}_1)^{-1}(S^2_-) &\overset{\cong}{\longleftrightarrow} S^2_- \times S^1 \\ [t; z, w] &\longmapsto \big([t, zw], w\big). \end{aligned}$$

This exhibits S^3 as an S^1 bundle over S^2. Using the long exact sequence on homotopy groups for a fibration, together with the fact that $\pi_3(S^1) = \pi_2(S^1) = 0$, we conclude that this map $J(\mathrm{id}_1)$ induces an isomorphism between

$$\pi_3(S^3) \xrightarrow{J(\mathrm{id}_1)_*} \pi_3(S^2),$$

which shows that $J(\mathrm{id}_1) \simeq \pm h$ the hopf map.

4 Appendix: Stability

The final property we wish to mention is the way in which J plays nicely with suspensions. This section only gives an indication of the desired stability properties, and not a complete proof. It is for this reason that we have relegated this section to an appendix.

Note that for every $n \geqslant 1$, there is an inclusion

$$\iota_n : O(n) \hookrightarrow O(n+1).$$

This can be realized as thinking of orthogonal transformations of $\mathbb{R}^n$ as simply the orthogonal transformations of $\mathbb{R}^{n+1}$ that fix the last coordinate. We can then go ahead and think of this as rotations(/reflections) of S^n that fix the north and south poles. Using the standard basis for $\mathbb{R}^{n+1}$, this map can be expressed in terms of matrices as

$$[A] \longmapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

If we identify $O(m)$ with it's inclusion into the homeomorphism group of S^{m-1}, for each m, and if we identify $\Sigma S^{n-1} \cong S^n$, then for any $A \in O(n)$, we see that

$$\iota_n(A) \;=\; \Sigma(A) : S^n \to S^n.$$

Note that

$$I \times I \times X \times Y \overset{q_1}{\twoheadrightarrow} S^k * \Sigma S^{n-1} \;\cong\; S^{k+n+1} \;\cong\; \Sigma\big(S^k * S^{n-1}\big) \overset{q_2}{\twoheadleftarrow} I \times I \times X \times Y$$

Applying the J construction yields

$$\begin{aligned} J(\iota_n \circ f)\Big(\big[t; x, [t', y]\big]\Big) &= \Big[t, \iota_n\big(f(x)\big)[t', y]\Big] \\ &= \Big[t, \Sigma\big(f(x)\big)[t', y]\Big] \\ &= \Big[t, [t', f(x)y]\Big] \\ &= \Big[t, J(f)\big([t'; x, y]\big)\Big] \\ &= \Sigma\big(J(f)\big)\Big(\Big[t, [t'; x, y]\Big]\Big). \end{aligned}$$

In other words, we have that

$$J(\iota_n \circ f) \circ q_1 \;=\; \Sigma J(f) \circ q_2.$$

We wish to show that there is actually a homeomorphism $S^k * \Sigma S^{n-1} \cong \Sigma\big(S^k * S^n\big)$ which will allow us to identify these two spaces in a way that allows for

$$J(\iota_n \circ f) \;\simeq\; \Sigma J(f).$$

Unfortunately, our calculation as it stands is not strong enough to imply the result, since the relation $[t; x, [t', y]] \leftrightarrow [t, [t'; x, y]]$ is not a function in either direction. Though it is true that the J-homomorphism can be shown to be stable (see for example [1]) in this sense, it is often merely a formal consequence of having used a more sophisticated definition.

The author is still searching for an elementary proof of this property, and would greatly appreciate a an email at scsanfor@iu.edu if you have any insight into this problem.

References

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