How Trivial is $\mathsf{Vec}_{\mathbb{K}}$?

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Abstract

The goal of this note is to show that $\mathsf{Vec}_{\mathbb{K}}$ (the category of finite dimensional vector spaces over a field $\mathbb{K}$) is trivial in many ways.

No Nontrivial Morphisms

Proposition 1 *The natural endomorphisms of the identity functor on* $\mathsf{Vec}_{\mathbb{K}}$ *correspond precisely to scalars.*

Proof: There exists a canonical, natural isomorphism

$$elts \,:\, \mathsf{Vec}_{\mathbb{K}}(\mathbb{K}, -) \;\to\; \mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}},$$

which identifies each vector space V with the vector space of its elements $\mathsf{Vec}_{\mathbb{K}}(\mathbb{K}, V)$. By precomposing with this natural isomorphism, then applying the Yoneda lemma[1], we obtain the desired result:

$$\begin{aligned}\mathbf{Nat}\big(\mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}, \mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}\big) &\cong \mathbf{Nat}(\mathsf{Vec}_{\mathbb{K}}(\mathbb{K}, -), \mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}) \\ &\cong \mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}(\mathbb{K}) \;=\; \mathbb{K}.\end{aligned}$$

□

[1] This kind of enriched Yoneda lemma does not work if we pass to division algebras, and in fact $\mathbf{Nat}\big(\mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}, \mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}\big) \cong Z(\mathbb{K})$ in general. The issue lies in the fact that noncommutative rings do not have a valid internal hom in the classical sense.

Proposition 2 *The group* $\mathbf{Aut}_{\otimes}\big(id_{\mathsf{Vec}_{\mathbb{K}}}\big)$ *of monoidal natural automorphisms of the identity functor on* $\mathsf{Vec}_{\mathbb{K}}$ *is trivial.*

Proof: Tracing through the isomorphism established in 1, we find that if $\eta^k : \mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}} \to \mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}$ is the natural transformation that corresponds to the scalar $k \in \mathbb{K}$, then for all $v \in V$,

$$\eta^k_V(v) \;=\; kv.$$

Since the tensorators of the identity functor are themselves identities, the statement that η^k be monoidal is precisely that

$$\eta^k_{U\otimes V} \;=\; \eta^k_U \otimes \eta^k_V.$$

In other words, for any $u \otimes v \neq 0$ we deduce that

$$\begin{aligned} ku \otimes v &= \eta^k_{U\otimes V}\big(u\otimes v\big) \\ &= \big(\eta^k_U \otimes \eta^k_V\big)\big(u\otimes v\big) \\ &= (ku)\otimes(kv) \\ &= k^2 u\otimes v \\ \therefore\;\; k &= k^2. \end{aligned}$$

Since $\mathbb{K}$ is a field, this implies that the only possible scalars that yield monoidal natural transformations are $k = 0, 1$. According to the conventions of [1], in order for η^k to be a monoidal natural transformation, $\eta^k_{\mathbb{1}}$ must be an isomorphism, and so the only possibility is $k = 1$. This restriction was necessary anyhow, since we wish only to consider monoidal natural isomorphisms, and $k = 0$ implies all components are the zero morphism. Thus we conclude that

$$\mathbf{Aut}_{\otimes}\big(\mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}\big) = \Big\{\mathrm{id}_{\mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}}\Big\}$$

□

Since the collection of all possible pivotal structures on $\mathsf{Vec}_{\mathbb{K}}$ is a torsor over this group we obtain:

Corollary 2.1 *There is a unique pivotal structure on* $\mathsf{Vec}_{\mathbb{K}}$*, given by*

$$\begin{aligned} \alpha : id_{\mathsf{Vec}_{\mathbb{K}}} &\longrightarrow \quad (-)^{**} \\ \alpha_V : V &\longrightarrow \quad V^{**} \\ v &\longmapsto \big(f \mapsto f(v)\big). \end{aligned}$$

In addition, this pivotal structure is spherical, and the quantum traces and dimensions agree with the classical traces and dimensions of $\mathsf{Vec}_{\mathbb{K}}$.

So far we have seen that the identity functor $\mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}}$ is a somewhat inflexible functor. Let us show that $\mathsf{Vec}_{\mathbb{K}}$ itself is inflexible.

Proposition 3 *All ($\mathbb{K}$-linear) autoequivalences $F : \mathsf{Vec}_{\mathbb{K}} \to \mathsf{Vec}_{\mathbb{K}}$ are naturally isomorphic to $id_{\mathsf{Vec}_{\mathbb{K}}}$.*

Proof: At first let $F : \mathsf{Vec}_{\mathbb{K}} \to \mathsf{Vec}_{\mathbb{K}}$ be any additive functor. In particular this means that it preserves direct sums. Since every vector space V is abstractly isomorphic to some direct sum of copies of $\mathbb{K}$, say $\mathbb{K}^{\oplus d}$, the image FV must be isomorphic to $(F\mathbb{K})^{\oplus d}$. In turn, $F\mathbb{K} \cong \mathbb{K}^{\oplus n}$ for some n, and thus $FV \cong \mathbb{K}^{\oplus nd}$. Effectively, this argument shows that F must scale dimensions by the factor n, and hence that every vector space in the image of F must have dimension dividing n.

Now suppose that F is an equivalence. Since F is essentially surjective, every vector space must be isomorphic to something in the image of F. By our previous observation, this implies that $n = 1$, and hence there is an isomorphism $\phi : \mathbb{K} \to F\mathbb{K}$. Furthermore, we know that $V \cong FV$ for every vector space, but the goal here is to try make this *natural*. Since F is fully faithful, we have that the map

$$\underline{F} : \mathsf{Vec}_{\mathbb{K}}\big(V, W\big) \to \mathsf{Vec}_{\mathbb{K}}\big(FV, FW\big)$$

which is natural in both V and W, is an isomorphism. Combining these facts, we get that

$$\begin{aligned} \mathsf{Vec}_{\mathbb{K}}\big(X, V\big) &\cong \mathsf{Vec}_{\mathbb{K}}\Big(X, \mathsf{Vec}_{\mathbb{K}}\big(\mathbb{K}, V\big)\Big) \\ &\cong \mathsf{Vec}_{\mathbb{K}}\Big(X, \mathsf{Vec}_{\mathbb{K}}\big(F\mathbb{K}, FV\big)\Big) \\ &\cong \mathsf{Vec}_{\mathbb{K}}\Big(X, \mathsf{Vec}_{\mathbb{K}}\big(\mathbb{K}, FV\big)\Big) \\ &\cong \mathsf{Vec}_{\mathbb{K}}\big(X, FV\big), \end{aligned}$$

where each isomorphism above is natural in both X and V. Applying the Yoneda Lemma, we obtain an isomorphism $V \to FV$ which is natural in V, or in other words, a natural isomorphism $\mathrm{id}_{\mathsf{Vec}_{\mathbb{K}}} \to F$. □

Propositions 1 and 3 imply the following important fact:

Corollary 3.1 *Any monoidal endofunctor* (F, J) *of* $\mathsf{Vec}_{\mathbb{K}}$ *is uniquely determined by the scalar* $J_{\mathbb{K},\mathbb{K}} : F(\mathbb{K} \otimes \mathbb{K}) \to F(\mathbb{K}) \otimes F(\mathbb{K})$*. Moreover if* $J_{\mathbb{K},\mathbb{K}} = k$*, then the natural transformation* $\eta_V : v \mapsto \frac{1}{k} \cdot v$ *is the unique monoidal isomorphism* $(F, J) \to (id_{\mathsf{Vec}_{\mathbb{K}}}, id_{-})$*.*

No Nontrivial Braidings

Clearly the standard swap of tensor factors

$$b_{X,Y} : X \otimes Y \ \ni \ x \otimes y \mapsto y \otimes x \ \in \ Y \otimes X$$

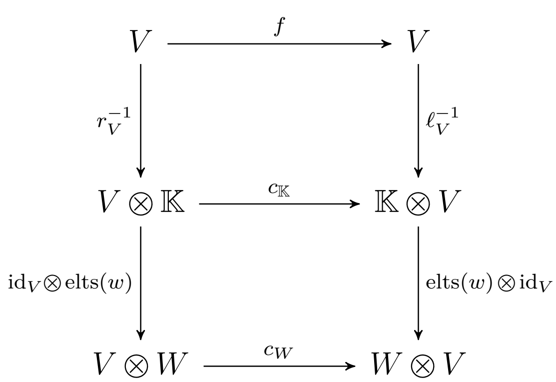
endows $\mathsf{Vec}_{\mathbb{K}}$ with a symmetric braiding, but are there others? It turns out that $\mathsf{Vec}_{\mathbb{K}}$ is boring on this front as well because this is the only braiding that it has. In fact, $\mathsf{Vec}_{\mathbb{K}}$ doesn't even have any half braidings!

Proposition 4 *Let* $V \in \mathsf{Vec}_{\mathbb{K}}$. *If* $c : (V \otimes -) \to (- \otimes V)$ *is a natural isomorphism satisfying the identity*

$$\forall X, Y \in \mathsf{Vec}_{\mathbb{K}}, \ \ c_{X \otimes Y} \ = \ \big(id_X \otimes c_Y\big) \circ \big(c_X \otimes id_Y\big)$$

i.e. *if* c *is a half-braiding for* V, *then* $c_X = b_{V,X}$.

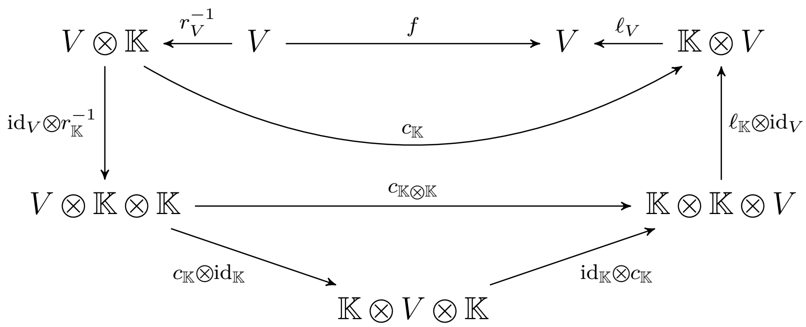
Proof: Using the isomorphism of elements from 1, we have that for any $w \in W \in \mathsf{Vec}_{\mathbb{K}}$, the map $\mathrm{elts}(w) : 1 \mapsto w$. This allows us to understand c_W in terms of $c_{\mathbb{K}}$. Consider the following commutative diagram:



Whatever the map $f : V \to V$ is, it is completely determined by $c_{\mathbb{K}}$ and *vice versa*. Using the diagram, we can calculate:

$$\begin{aligned} c_W(v \otimes w) &= \big(c_W \circ (\mathrm{id}_V \otimes \mathrm{elts}(w)) \circ r_V^{-1}\big)(v) \\ &= \big((\mathrm{elts}(w) \otimes \mathrm{id}_V) \circ \ell_V^{-1} \circ f\big)(v) \\ &= \big(\mathrm{elts}(w) \otimes \mathrm{id}_V\big)\big(1 \otimes f(v)\big) \\ &= w \otimes f(v). \end{aligned}$$

This shows that c_W doesn't do anything to W, and simply swaps factors while applying a fixed map f to the V factor. Using this characterization, consider another commutative diagram:



In the above, we have used the fact that c is natural, and a half-braiding, as well as the fact that $r_{\mathbb{K}} = \ell_{\mathbb{K}}$ (see [1], Cor. 2.2.5). Following the map around the outer path yields f^2, but commutativity of the diagram implies this is the same as just f. Since $f : V \to V$ is an isomorphism, $f^2 = f$ implies $f = \mathrm{id}_V$, and this proves the claim. □

Since all objects admit a unique half-braiding, we obtain:

Corollary 4.1 *There exists a canonical isomorphism of (braided, monoidal) categories*

$$\mathsf{Vec}_{\mathbb{K}} \to \mathcal{Z}(\mathsf{Vec}_{\mathbb{K}})$$
$$V \mapsto \big(V\,,\, b_{V,-}\big),$$

where $\mathcal{Z}(-)$ denotes the Drinfel'd center.

Also, since every braiding induces a half braiding by fixing one of the factors, we obtain:

Corollary 4.2 *The symmetric swap $b_{X,Y}$ is the one and only braiding that exists on $\mathsf{Vec}_{\mathbb{K}}$.*

References

[1] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.