A Trace on FinSet

Sean Sanford

Abstract

In these notes we calculate the coend

$$\int^{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X),$$

along with its natural map

$$\tau : \coprod_{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X) \to \int^{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X),$$

which we interpret as a kind of trace. It turns out that this object is conveniently described as the set P of all partitions of all natural numbers.

Sections 1 through 5 are dedicated to proving the main theorem, and section 6 follows up with some elementary applications. In section 7 we describe a semiring structure on P that is related to τ. We go on to use this structure in section 8 to analyze the category of finite G-sets in close analogy with classical representation theory.

1 The Cowedge Condition and the Relation $\sim$

Here we give a specific interpretation of the idea of coends that is relevant to our situation. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $K : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor.

Definition. A morphism of the form

$$\eta : \coprod_{C \in \mathcal{C}} K(C, C) \to D$$

is called a *cowedge* under K if for all $f : C \to C'$ the following square commutes:

$$\begin{array}{ccc} K(C', C) & \xrightarrow{K(f,C)} & K(C, C) \\ {\scriptstyle K(C',f)}\downarrow & & \downarrow{\scriptstyle \eta} \\ K(C', C') & \xrightarrow{\quad \eta \quad} & D \end{array}$$

In these notes, we will be concerned with the situtation where $\mathcal{C} = \mathsf{FinSet}$, $\mathcal{D} = \mathsf{Set}$ and $K(\, , \,) = \mathsf{FinSet}(\, , \,)$. In this situation, $\mathsf{FinSet}(f, C)$ is usually written as f^* and is precomposition with f, while $\mathsf{FinSet}(C', f)$ is usually written f_* and is postcompostion with f. By making the appropriate substitutions to the above diagram, we arrive at

$$\begin{array}{ccc} \mathsf{FinSet}(B, A) & \xrightarrow{f^*} & \mathsf{FinSet}(A, A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \eta} \\ \mathsf{FinSet}(B, B) & \xrightarrow{\quad \eta \quad} & S \end{array}$$

Note that if $g \in \mathsf{FinSet}(B, A)$, then commutativity of the diagram implies $\eta(g \circ f) = \eta(f \circ g)$. This will be an important formula for us, so we give it a name.

Definition. For the sake of these notes, a morphism

$$\eta : \coprod_{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X) \to S$$

is said to satisfy the *cowedge condition* if $\eta(g \circ f) = \eta(f \circ g)$ whenever both compositions are defined.

Thus we find that such a morphism out of the coproduct is a cowedge under $\mathsf{FinSet}(\, , \,)$ if and only if it satisfies the cowedge condition.

Definition. A *coend* is an initial cowedge. This means that τ is a coend of K if for every cowedge η under K there is a unique map ζ such that $\eta = \zeta \circ \tau$. By abuse of notation the term coend also refers to the object at the codomain of the cowedge, and for this object we use the notation

$$\int^{C \in \mathcal{C}} K(C, C),$$

where the symbol C acts as an index or 'dummy variable' similar to the notation used in products and coproducts.

The conclusion of these notes is that

$$\int^{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X) \cong P := \{\text{all partitions of all natural numbers}\}.$$

First we begin by examining the consequences of the cowedge condition. If η satisfies the cowedge condition, then η maps $f \circ g$ and $g \circ f$ to the same element. This suggests investigation of the relation $\sim_0$ where $\phi \sim_0 \psi$ if there exist f, g such that $\phi = f \circ g$ and $\psi = g \circ f$. This relation is reflexive and symmetric, but it is not transitive, and thus not an equivalence relation. However there is a canonical way to fixing this issue.

Definition. The relation $\sim$ on $\coprod_{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X)$ is defined to be the transitive closure of $\sim_0$. This means that $\phi \sim \psi$ if there exists a sequence of morphisms $(f_i)_{i=1}^k$ such that $\phi \sim_0 f_1$, $f_i \sim_0 f_{i+1}$ for $i < k$ and $f_k \sim_0 \psi$.

with this definition, it is not difficult to see that for any η satisfying the cowedge condition, $f \sim g$ implies $\eta(f) = \eta(g)$.

2 Reduction to Isomorphisms

Using the epi-monic factorization in FinSet, we can write any map $f \in \mathsf{FinSet}(A, A)$ as $f = m \circ e$ where $e : A \twoheadrightarrow \mathrm{im}(f)$ and $m : \mathrm{im}(f) \hookrightarrow A$. Knowing that f can be decomposed in this way tells us that $f \sim e \circ m$, and this map $e \circ m \in \mathsf{FinSet}(\mathrm{im}(f), \mathrm{im}(f))$ is $f|_{\mathrm{im}(f)}$. This allows us to prove our first proposition.

Proposition 2.1. *For every endomorphism f in* FinSet*, there is an isomorphism $\hat{f}$ such that $f \sim \hat{f}$.*

The proof of this proposition will first require a lemma:

Lemma 2.1.1. *For every endomorphism $f \in \mathsf{FinSet}(A, A)$ in FinSet there is a natural number $N \in \mathbb{N}$ such that $f|_{im(f^n)} = f|_{im(f^N)}$ for all $n \geq N$. Moreover, this common map is an isomorphism, and if $A \neq \emptyset$, then $im(f^N) \neq \emptyset$.*

Proof of lemma. Firstly, for $f : A \to A$, we have that $\mathrm{im}(f) \subseteq A$. This implies that $\mathrm{im}(f^2) = f(\mathrm{im}(f)) \subseteq f(A) = \mathrm{im}(f)$, and by induction, we have a descending sequence of subsets:

$$\cdots \subseteq \mathrm{im}(f^{k+1}) \subseteq \mathrm{im}(f^k) \subseteq \cdots \subseteq \mathrm{im}(f^2) \subseteq \mathrm{im}(f) \subseteq A$$

If $A = \emptyset$, then f was an isomorphism to begin with, and the sequence above is constant. Assume then that $A \neq \emptyset$. Since the image of a nonempty set is nonempty, every term in this sequence will always have at least one element.

If it happens that $f|_{\mathrm{im}(f^N)}$ is an isomorphism, then $\mathrm{im}(f^{N+1}) = f(\mathrm{im}(f^N)) = \mathrm{im}(f^N)$, so by induction $\mathrm{im}(f^n) = \mathrm{im}(f^N)$ for all $n \geq N$.

For the sake of contradiction, suppose that for all n, $f|_{\mathrm{im}(f^n)}$ is not an isomorphism. Since A is finite, $f|_{\mathrm{im}(f^n)}$ is not surjective for any n. Thus we have that $\#(\mathrm{im}(f)) \leq \#(A) - 1$, and by induction $\#(\mathrm{im}(f^n)) \leq \#(A) - n$. If $\#(A) = N$, then $\#(\mathrm{im}(f^N)) \leq \#(A) - N = 0$, which contradicts our previous observation that $1 \leq \#(\mathrm{im}(f^N))$ ↯ .

□

We can now proceed to prove Proposition 2.1.

Proof of Proposition. Given $f \in \mathsf{FinSet}(A, A)$, define

$$\rho(f) := f|_{\mathrm{im}(f)} \in \mathsf{FinSet}\big(\mathrm{im}(f), \mathrm{im}(f)\big).$$

We have already seen that $f \sim \rho(f)$, and this implies that $\rho(f) \sim \rho(\rho(f)) =: \rho^2(f)$. Since $\sim$ is transitive, $f \sim \rho^2(f)$ and by iduction $f \sim \rho^n(f)$ for all $n \geq 1$. By the lemma, it will suffice to prove that $\rho^n(f) = f|_{\mathrm{im}(f^n)}$ for all $n \geq 1$.

Note that $\rho(f) := f|_{\mathrm{im}(f^1)}$, so our base case is covered by definition. Suppose that $\rho^k(f) = f|_{\mathrm{im}(f^k)}$ for all $k \leq n$. We calculate

$$\begin{aligned}
\rho^{n+1}(f) :&= \rho\left(\rho^n(f)\right) \\
&= \rho\left(f|_{\mathrm{im}(f^n)}\right) \\
&= \left(f|_{\mathrm{im}(f^n)}\right)|_{\mathrm{im}\left(f|_{\mathrm{im}(f^n)}\right)} \\
&= \left(f|_{\mathrm{im}(f^n)}\right)|_{f|_{\mathrm{im}(f^n)}(\mathrm{im}(f^n))} \\
&= \left(f|_{\mathrm{im}(f^n)}\right)|_{f(\mathrm{im}(f^n))} \\
&= \left(f|_{\mathrm{im}(f^n)}\right)|_{\mathrm{im}(f^{n+1})} \\
&= f|_{\mathrm{im}(f^{n+1})}.
\end{aligned}$$

Thus by induction $\rho^n(f) = f|_{\mathrm{im}(f^n)}$ for all n. Using the lemma, define $\hat{f} := \rho^N(f)$ where N is the first number such that $f|_{\mathrm{im}(f^N)}$ is an isomorphism. □

3 The Maps τ_0 and τ

Here we describe some important morphisms between relevent objects, and examine their properties. For any $f \in \mathsf{Aut}_{\mathsf{FinSet}}(A)$, the subgroup $\langle f \rangle$ generated by f is a cyclic group that acts on A. Consider the orbit space

$$A/\langle f \rangle.$$

By ordering the sizes of the orbits from greatest to least, we obtain a partition of the number $|A|$. Let the partition thus obtained from f be denoted $\tau_0(f)$. We have just described a map

$$\tau_0 : \coprod_{X \in \mathsf{FinSet}} \mathsf{Aut}(X) \to P := \{\text{all partitions of all natural numbers}\}.$$

By the universal property of the coproduct, the inclusions $i_X : \mathsf{Aut}(X) \hookrightarrow \mathsf{FinSet}(X, X)$ form a cocone over the summands, and determine a unique map

$$i : \coprod_{X \in \mathsf{FinSet}} \mathsf{Aut}(X) \to \coprod_{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X),$$

which is i_X on the X component. The construction of Proposition 2.1 shows that we have a surjection $r : f \mapsto \hat{f}$ which goes in the other direction, and satisfies $r \circ i = \mathrm{id}$.

We now make what is possibly the most important definition in these notes.

Definition. The *trace map* τ is the composition $\tau_0 \circ r$. Given any map $f \in \mathsf{FinSet}(A, A)$ the partition $\tau(f) \in P$ is called the *trace* of f.

Proposition 3.1. *The trace map τ is a cowedge under the bifunctor* $\mathsf{FinSet}(\,,\,)$.

Proof of Proposition. It will suffice to prove that $\tau_0 \circ r$ satisfies the cowedge condition. Suppose then that $f : A \to B$ and $g : B \to A$ are any two functions of the finite sets A and B. By Lemma 2.1.1 there are integers N_1 and N_2 such that

$$r(f \circ g) = (f \circ g)|_{\mathrm{im}((f\circ g)^{N_1})}, \quad \& \\ r(g \circ f) = (g \circ f)|_{\mathrm{im}((g\circ f)^{N_2})}.$$

Now set

$$N := \max\{N_1, N_2\} \\ A_0 := \mathrm{im}\big((g \circ f)^N\big) \\ B_0 := \mathrm{im}\big((f \circ g)^N\big).$$

We calculate:

$$\begin{aligned} B_0 &= (f \circ g)^N(B) \\ &= (f \circ g)^{N+1}(B) \\ &= \big(f \circ (g \circ f)^N \circ g\big)(B) \\ &= \big(f \circ (g \circ f)^N\big)\big(g(B)\big) \\ &\subseteq \big(f \circ (g \circ f)^N\big)(A) \\ &= \big((f \circ g)^N \circ f\big)(A) \\ &= (f \circ g)^N\big(f(A)\big) \\ &\subseteq (f \circ g)^N\big(B\big) = B_0 \\ &\Longrightarrow \\ B_0 &= \big(f \circ (g \circ f)^N\big)(A) \;=\; f(A_0), \end{aligned}$$

which shows that $f|_{A_0} : A_0 \to B_0$ is surjective. We also have that

$$g|_{B_0} \circ f|_{A_0} \;=\; (g \circ f)|_{A_0} \;=\; r(g \circ f) \;\text{ is iso,}$$

and this forces $f|_{A_0}$ to be injective and hence an isomorphism. Similar arguments show that $g|_{B_0}$ is also an isomorphism. For brevity let us denote these maps f_0 and g_0.

Now, notice that

$$\begin{aligned} r(f \circ g) &= \widehat{f \circ g} \;=\; f_0 \circ g_0, \;\; \& \\ r(g \circ f) &= \widehat{g \circ f} \;=\; g_0 \circ f_0. \end{aligned}$$

This implies that we have $r(f \circ g)^k = \mathrm{id}_{B_0}$ if and only if $r(g \circ f)^k = \mathrm{id}_{A_0}$. This fact then tells us that $\langle r(f \circ g) \rangle \cong \langle r(g \circ f) \rangle$, so let us denote this common cyclic group by $G = \langle z \rangle$. Interpreting A_0 and B_0 as G-sets shows us another important fact:

$$\begin{aligned} z.f_0(a) &= r(f \circ g)\big(f_0(a)\big) \\ &= (f_0 \circ g_0)\big(f_0(a)\big) \\ &= f_0\big((g_0 \circ f_0)(a)\big) \\ &= f_0\big(r(g \circ f)(a)\big) \\ &= f_0(z.a) \\ &\therefore \;\; f_0 \text{ is equivariant!} \end{aligned}$$

Since f_0 is an equivariant isomorphism, it follows that A_0 and B_0 are isomorphic as G-sets, and hence have identical orbit structure. In light of this, we conclude that

$$(\tau_0 \circ r)(f \circ g) = (\tau_0 \circ r)(g \circ f),$$

which is precisely the cowedge condition. □

4 The Splitting σ of τ

For every $n \in \mathbb{N}$, there is a set $\underline{n} := \{1, \ldots, n\}$ (and $\underline{0} := \emptyset$). The symmetric groups are defined to be the automorphism groups of these sets: $S_n := \mathsf{Aut}(\underline{n})$. The disjoint union of these S_n form an important subset:

$$\coprod_{n=0}^{\infty} S_n \overset{j}{\hookrightarrow} \coprod_{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X).$$

whose inclusion map we denote by j when necessary.

Given a partition $I \in P$ of n, we will construct a permutation $\sigma(I) \in S_n$ that satisfies $\tau\big(\sigma(I)\big) = I$. Assuming the partation is given as $I = (n_1, n_2, \cdots, n_r)$ with $n_i \geq n_{i+1}$, set $n_0 = 0$ and define

$$\sigma(I) = \prod_{k=0}^{r-1} \left(1 + \sum_{i=1}^{k} n_i \quad 2 + \sum_{i=1}^{k} n_i \quad \ldots \quad n_{k+1} + \sum_{i=1}^{k} n_i \right).$$

Example 4.1. If $I = (3, 2, 2, 1)$, then $\sigma(I) = (1\ 2\ 3)(4\ 5)(6\ 7)(8) \in S_8$.

Proposition 4.1. *The map $\tau \circ \sigma = id_P$, and for any $\nu \in S_n$, there is some $\gamma \in S_n$ such that*

$$(\sigma \circ \tau)(\nu) = \gamma \circ \nu \circ \gamma^{-1}.$$

Proof of Proposition. The first statement is by construction. The second follows from the classical result of finite group theory that any two permutations are conjugate if and only if they have the same cycle decomposition. □

5 Verification of the Universal Property

Before proving that τ satisfies the universal property of the coend, we will need a lemma.

Lemma 5.0.1. *If two cowedges agree on the subset $\coprod_n S_n$, then they are equal.* i.e. *for any two cowedges η and ζ,*

$$\eta \circ j = \zeta \circ j \qquad \Longrightarrow \qquad \eta = \zeta.$$

Proof of Lemma. Suppose $\eta(\nu) = \zeta(\nu)$ for all $\nu \in \coprod_n S_n$. Let $f \in \mathsf{FinSet}(A, A)$ be any endomorphism of a finite set A. By Proposition 2.1, $\hat{f}$ is an automorphism of $A_0 := \operatorname{im}(f^N)$ for some N. If $\#(A_0) = k$, then there is some bijection $\phi : A_0 \to \underline{k}$. Note that

$$f \ \sim\ \hat{f} \ =\ \left(\hat{f} \circ \phi^{-1}\right) \circ \phi \ \sim\ \phi \circ \hat{f} \circ \phi^{-1} \ \in\ S_k.$$

Using this, we see that

$$\eta(f) \ =\ \eta\big(\phi \circ \hat{f} \circ \phi^{-1}\big) \ =\ \zeta\big(\phi \circ \hat{f} \circ \phi^{-1}\big) \ =\ \zeta(f).$$

Since f was arbitrary, the two cowedges must be equal. □

Theorem 5.1. *The map*

$$\tau : \coprod_{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X) \ \to \ P$$

satisfies the universal property, and therefore

$$\int^{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X) \ \cong P.$$

Proof of Theorem. Let η be a cowedge under $\mathsf{FinSet}(\ ,\ )$. Define $\xi := \eta \circ \sigma$. We will show that $\eta = \xi \circ \tau$ and that any other map with this property is equal to ξ. Firstly suppose that $\nu \in S_k$. By Lemma 4.1, there is some $\gamma \in S_k$ such that $(\sigma \circ \tau)(\nu) = \gamma \circ \nu \circ \gamma^{-1}$. Using this and the cowedge condition, we calculate:

$$\begin{aligned} (\xi \circ \tau)\,(\nu) &= \eta\big((\sigma \circ \tau)(\nu)\big) \\ &= \eta\left(\gamma \circ \nu \circ \gamma^{-1}\right) \\ &= \eta(\nu). \end{aligned}$$

Since $\eta(\nu) = (\xi \circ \tau)(\nu)$ and $\nu \in S_k$ was an arbitrary permutation, by Lemma 5.0.1 it must be the case that $\eta = \xi \circ \tau$.

Secondly, suppose there was some other ζ such that $\zeta \circ \tau = \eta = \xi \circ \tau$. By Proposition 4.1 we find that

$$\zeta = \zeta \circ \mathrm{id}_P = \zeta \circ \tau \circ \sigma = \eta \circ \sigma = \xi \circ \tau \circ \sigma = \xi \circ \mathrm{id}_P = \xi.$$

Thus every cowedge under $\mathsf{FinSet}(\ ,\ )$ factors uniquely through τ, which is the universal property we set out to prove. Finally, note that the uniqueness of the factorization implies that the object $\int^{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X)$ is unique up to unique isomorphism, and we can conclude that

$$P \underset{\text{canonical}}{\cong} \int^{X \in \mathsf{FinSet}} \mathsf{FinSet}(X, X).$$

□

6 Some Applications

By the universal property of the coend, any function with domain P will yield an invariant of endomorphisms in FinSet, and all such invariants that satisfy the cowedge condition arise in this way.

Example 6.1. Let $F_1 : P \to \mathbb{N}$ be the map defined by

$$F_1(I) = \text{the total number of 1s appearing in } I.$$

Using this function we obtain the invariant $F_1 \circ \tau$ which sends f to the number of fixed points of f.

Generalizing this, we obtain

Example 6.2. Let $F_n : P \to \mathbb{N}$ be the map defined by

$$F_n(I) = \text{the total number of } n\text{s appearing in } I.$$

Using this function we obtain the invariant $F_n \circ \tau$ which sends f to the number of distinct strict n-cycles of points of f, *i.e.* the number of disjoint subsets $\{x_i\}_{i=1}^n$ of $\operatorname{dom}(f)$ on which f acts as an n-cycle. By multiplying this quantity by n we obtain $n \cdot (F_n \circ \tau)(f)$ the number of strictly n-periodic points of f.

Example 6.3. Let $T : P \to \mathbb{N}$ be the map defined by

$$T(I) = \text{the sum of all terms appearing in } I.$$

Using this function we obtain the invariant $T \circ \tau$ which sends f to the number of periodic points of f. Of course, since f is an endomorphism of a finite set, the collection of all periodic points is precisely the largest subset of $\operatorname{dom}(f)$ on which f is an isomorphism. Thus $(T \circ \tau)(f)$ is exactly the cardinality of this set. Since every periodic point of f is strictly n-periodic for some $n \geq 1$, we have that

$$(T \circ \tau)(f) \;=\; \sum_{n=1}^{\infty} n \cdot (F_n \circ \tau)(f),$$

where the sum is well defined, because all but finitely many terms are zero.

The set $\coprod_X \mathsf{FinSet}(X, X)$ has a natural commutative monoid structure coming from the operation $(f, g) \mapsto f \sqcup g$. The set P has a natural commutative monoid structure arising from the operation of juxtaposition $(I, J) \mapsto I + J$. The partition $I + J$ has all the terms from I and J together, rearranged if necessary. Using the techniques we have developed, it is not hard to show that τ acts as a homomorphism of these monoids, *i.e.* that

$$\tau(f \sqcup g) = \tau(f) + \tau(g).$$

7 A Semiring Structure on P

Here we exposit on further algebraic structure that exists on P, namely that of multiplication, and discuss how it interacts with juxtaposition to endow P with the structure of a commutative, unital semiring.

Definition. For the remainder of the text, for every $n \in \mathbb{N}$ with $n \geq 1$ we will use the notation

$$z_n = (n) \in P$$

for the partition of n consisting of just n itself.

The empty partition acts as an identity with respect to juxtaposition, and so we will prefer to abbreviate it as 0_P or simply 0 when no confusion should arise.

With this notation in place, note that any partition can be written as a finite sum of the z_n:

Example 7.1.

$$(5, 3, 2, 2, 2, 1, 1) = z_5 + z_3 + 3z_2 + 2z_1.$$

Given any two partitions I and $J \in P$, define the product partition $I \cdot J$ to be

Definition.

$$I \cdot J := \tau\big(\sigma(I) \times \sigma(J)\big).$$

In order to get a better sense of what this definition implies, we carry out the following calculation:

Proposition 7.1. *Let $\ell = lcd(n, m)$ and $g = gcd(n, m)$ be the least common multiple and greatest common divisor respectively of n and m. The product of z_n and z_m is given by*

$$z_n \cdot z_m \; = \; g\, z_\ell.$$

Proof. Directly from the definition, we have

$$\begin{aligned} z_n \cdot z_m :&= \tau\big(\sigma(z_n) \times \sigma(z_m)\big) \\ &= \tau\big((1\ 2\ \cdots\ n) \times (1\ 2\ \cdots\ m)\big). \end{aligned}$$

It will suffice to show that the cycle decomposition of this product map consists of g disjoint ℓ-cycles. Without loss of generality, assume that $n \leq m$. The first cycle will begin at the point $(1, 1)$ and must return to this point. In other words, the last point in the cycle will be (n, m). The very first time that the cycle will arrive at (n, m) will be at the ℓ^{th} step, so this must be an ℓ-cycle.

Now suppose that $0 < k < g$. If the point $(1, 1 + k)$ is in this first cycle, then there is some multiple tn of n such that

$$\begin{gathered} tn \equiv k \mod m \\ \Longrightarrow \\ \exists s \in \mathbb{Z}, \;\; tn = k + sm \\ \Longrightarrow \\ tn - sm = k \\ \Longrightarrow \\ g \mid k \;\; \lightning \end{gathered}$$

Thus each of the points $(1, 1 + k)$ for $0 < k < g$ lies outside of the first cycle. In fact, this argumet shows that all of these points lie in distinct cycles. From here it is easy to see that each of these must be ℓ-cycles and that these account for all points in $\underline{n} \times \underline{m}$. Thus we have

$$\begin{aligned} z_n \cdot z_m &:= \tau\big((1\ 2\ \cdots\ n) \times (1\ 2\ \cdots\ m)\big) \\ &= \tau\left(\bigsqcup_{k=0}^{g-1} \ell\text{-cycle containing } (1, 1+k)\right) \\ &= \sum_{k=0}^{g-1} \tau\big(\ell\text{-cycle containing } (1, 1+k)\big) \\ &= \sum_{k=0}^{g-1} (\ell) \\ &= g\, z_\ell. \end{aligned}$$

□

Note. In the category FinSet, cartesian products distribute over coproducts, and thus Proposition 7.1 completely determines the product structure on P. In particular, it shows that $z_1 = 1_P$.

Corollary 7.1.1. *As a commutative semiring,* $(P, +, 0_P, \cdot, 1_P)$ *is isomorphic to*

$$P \;\cong\; \mathbb{N}[z_2, z_3, z_4, \ldots]/\sim$$

where the relation $\sim$ *is generated by* $z_n \cdot z_m \sim g\, z_\ell$ *as dictated by Proposition 7.1.*

Example 7.2.

$$\begin{aligned} (4,3,3,1) \cdot (5,2,2,2) &= \big(z_4 + 2z_3 + 1\big) \cdot (z_5 + 3z_2) \\ &= z_{20} + 2z_{15} + z_5 + 6z_4 + 6z_6 + 3z_2 \\ &= z_{20} + 2z_{15} + 6z_6 + z_5 + 6z_4 + 3z_2 \\ &= (20, 15, 15, 6, 6, 6, 6, 6, 6, 5, 4, 4, 4, 4, 4, 4, 2, 2, 2). \end{aligned}$$

8 Applications to Finite G-Sets

Let $\rho : G \to \mathrm{Aut}(X)$ be an action of a finite group G on a finite set X. Using our huristic sense that τ is a kind of trace, we follow representation theory and define the *character* $\chi_X : G \to P$ of X to be

Definition.

$$\chi_X(g) := \tau\big(\rho(g)\big).$$

Since τ satisfies the cowedge contition, it is immediate that χ_X is a class function (it is constant on conjugacy classes). As it turns out, we can say much more:

Theorem 8.1. *The map $\chi : [X] \mapsto \chi_X$ defines a unital semiring homomorphism from the semiring of isomorphism classes of finite G-sets $\mathcal{R}$ to the semiring P^G_{class} of class functions from G to P.*

Proof. In order to begin making sense of this statement, it is necessary to understand the semiring structure on $\mathcal{R}$ and P^G_{class}. The latter is simply defined by pointwise addition and multiplication, so we will focus on the former. Since the product and coproduct of G-sets are respectively unique up to unique isomorphism, the definitions

$$\begin{gathered}[X] + [X'] := [X \sqcup X'], \text{ and} \\ [X] \cdot [X'] := [X \times X']\end{gathered}$$

endow $\mathcal{R}$ with a well-defined product and sum. The zero object $0_{\mathcal{R}}$ is easily seen to be the class $[\emptyset]$ with its obvious G-action, and the multiplicative identity $1_{\mathcal{R}}$ is the class of the G-set consisting of a single point with the trivial action. Verification of distributivity follows from distributivity for actual G-sets and is purely formal.

Now suppose that $\varphi : X \to X'$ is an isomorphism of G-sets. Then we have that for any $g \in G$,

$$\begin{aligned}\chi_X(g) &= \tau\big(\rho(g)\big) \\ &= \tau\big(\varphi \circ \rho(g) \circ \varphi^{-1}\big) \\ &= \tau\big(\rho'(g)\big) \\ &= \chi_{X'}(g).\end{aligned}$$

Thus $\chi : [X] \mapsto \chi_X$ is well-defined.

If $\nu : G \to \mathrm{Aut}(Y)$ is another G-action, then

$$g \mapsto \Big(\rho(g) \sqcup \nu(g) : X \sqcup Y \to X \sqcup Y\Big)$$

defines an action $\rho \sqcup \nu : G \longrightarrow \mathrm{Aut}(X \sqcup Y)$, and we have

$$\begin{aligned}\chi_{X\sqcup Y}(g) &= \tau\big(\rho(g) \sqcup \nu(g)\big)\\ &= \tau\big(\rho(g)\big) + \tau\big(\nu(g)\big)\\ &= \chi_X(g) + \chi_Y(g).\end{aligned}$$

We can also construct the product

$$g \mapsto \Big(\rho(g) \times \nu(g) : X \times Y \longrightarrow X \times Y\Big),$$

which defines an action $\rho \times \nu : G \longrightarrow \mathrm{Aut}(X \times Y)$. Using the definition of multiplication in P together with Proposition 4.1, we find that

$$\begin{aligned}\chi_{X\times Y}(g) &= \tau\big(\rho(g) \times \nu(g)\big)\\ &= \tau\left(\gamma_1^{-1} \circ \sigma\Big(\tau\big(\rho(g)\big)\Big) \circ \gamma_1 \times \gamma_2^{-1} \circ \sigma\Big(\tau\big(\nu(g)\big)\Big) \circ \gamma_2\right)\\ &= \tau\left(\big(\gamma_1 \times \gamma_2\big)^{-1} \circ \left(\sigma\Big(\tau\big(\rho(g)\big)\Big) \times \sigma\Big(\tau\big(\nu(g)\big)\Big)\right) \circ \big(\gamma_1 \times \gamma_2\big)\right)\\ &= \tau\left(\sigma\Big(\tau\big(\rho(g)\big)\Big) \times \sigma\Big(\tau\big(\nu(g)\big)\Big)\right)\\ &= \tau\big(\rho(g)\big) \cdot \tau\big(\nu(g)\big)\\ &= \chi_X(g) \cdot \chi_Y(g).\end{aligned}$$

In other words, we have that

$$\begin{aligned}\chi\big([X] + [Y]\big) &= \chi\big([X]\big) + \chi\big([Y]\big), \text{ and}\\ \chi\big([X] \cdot [Y]\big) &= \chi\big([X]\big) \cdot \chi\big([Y]\big).\end{aligned}$$

The fact that χ takes $1_{\mathcal{R}} \mapsto \big(g \mapsto 1_P\big)$ and $0_{\mathcal{R}} \mapsto \big(g \mapsto 0_P\big)$ are easy to verify, and this completes the proof.

□

Let us apply this theory to the group S_3 to show what the equivalent of a character table would be in this setting. The first thing to notice is that in this setting, complete reducibility or *semisimplicity* is the statement that every finite G-set can be written uniquely as a disjoint union of orbits. An orbit is just a transitive G-set, and these are the irreducible objects in the category of G-sets. Every orbit is necessarily of the form G/H for some subgroup H. In the case $G = S_3$, there are only 4 isomorphism classes of orbits and they are given below:

	id	$[(1\,2)]_{\text{conj}}$	$[(1\,2\,3)]_{\text{conj}}$
$\mathbb{1} \cong [S_3/S_3]$	1	1	1
$X := [S_3/\langle(1\,2\,3)\rangle]$	2	z_2	2
$Y := [S_3/\langle(1\,2)\rangle]$	3	$1 + z_2$	z_3
$S := [S_3]$	6	$3z_2$	$2z_3$

In the above table, the rows correspond to (isomorphism classes of) orbits and the columns correspond to conjugacy classes of elements of S_3. The entries are the values of the character of the row applied to any element of the conjugacy class of that column. Thanks to Theorem 8.1, by multiplying entries vertically we obtain the characters of the product G-sets. Here are some examples:

	id	$[(1\,2)]_{\text{conj}}$	$[(1\,2\,3)]_{\text{conj}}$
$X \cdot Y$	6	$3z_2$	$2z_3$
$X^2 = X \cdot X$	4	$2z_2$	4
$Y^2 = Y \cdot Y$	9	$1 + 4z_2$	$3z_3$

By semisimplicity, the set $\{\mathbb{1}, X, Y, S\}$ forms an $\mathbb{N}$-basis for $\mathcal{R}$, and we can use this basis together with the above tables to find that

$$XY = S$$
$$X^2 = 2X$$
$$Y^2 = Y + S.$$

The above shows that $\mathcal{R}$ is isomorphic to a quotient of $\mathbb{N}[X, Y]$. Specifically, we have

Theorem 8.2. *For $G = S_3$, the semiring $\mathcal{R}$ of isomorphism classes of G-sets is*

$$\mathcal{R} \cong \mathbb{N}[X, Y] \big/ \left\langle X^2 = 2X \,,\, Y^2 = XY + Y \right\rangle$$

where the angle brackets denote the congruence relation generated by these relations.