Partitions and Universal Traces

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http://pages.iu.edu/~scsanfor/

Introduction

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Main Idea

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- The property $\tau(f \circ g) = \tau(g \circ f)$ completely characterizes the trace of linear maps.

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Main Idea

- The property $\tau(f \circ g) = \tau(g \circ f)$ completely characterizes the trace of linear maps.
- In the categoy of finite sets this equation gives rise to a trace-like invariant that *assigns a partition to every endomorphism*.

Universal Traces

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Definition (Alt.)

The trace $\mathrm{tr}(-)$ is an invariant of endomorphisms in **Vec** that satisfies the equation $(*)$, *and is initial with respect to this property*.

Philosophizing

Universal Traces

That definition is so simple!

Perhaps this is why the trace is so important... 🤔

Does it have to be **Vec**?

Let's do **FinSet**

Universal Traces

Definition

The (universal) trace $\tau(-)$ is an invariant of endomorphisms in **FinSet** that satisfies $\tau(f \circ g) = \tau(g \circ f)$, *and is initial with respect to this property*.

The Case of Finite Sets

Observations

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- for an iso h, $\tau(f) = \tau\big(h \circ (h^{-1} \circ f)\big) = \tau(h \circ f \circ h^{-1})$
- Every endomorphism $f\colon X \to X$ can be split as a surjection $s : X \twoheadrightarrow \mathrm{im}(f)$ followed by an injection $i : \mathrm{im}(f) \hookrightarrow X$.

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$$\tau(f) \;=\; \tau(i \circ s)$$

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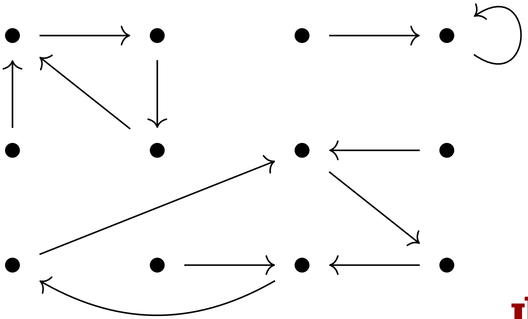
$$\tau(f) = \tau(i \circ s) = \tau(s \circ i) = \tau\big(f|_{\mathrm{im}(f)}\big)$$

How to Calculate

The Case of Finite Sets

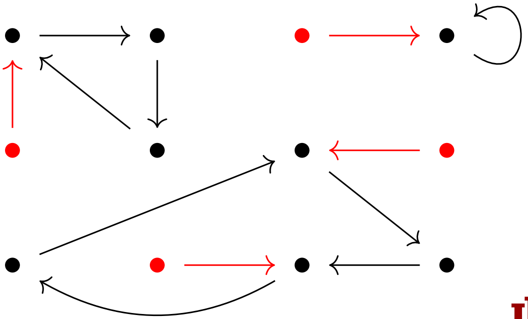
1. Take an endomorphism of a finite set.



How to Calculate

The Case of Finite Sets

1. Take an endomorphism of a finite set.
2. Restrict it to the largest subset on which it acts by an isomorphism.



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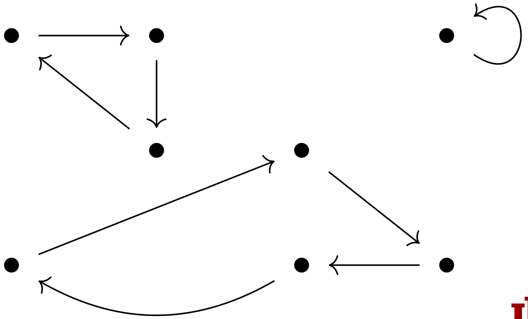

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1. Take an endomorphism of a finite set.
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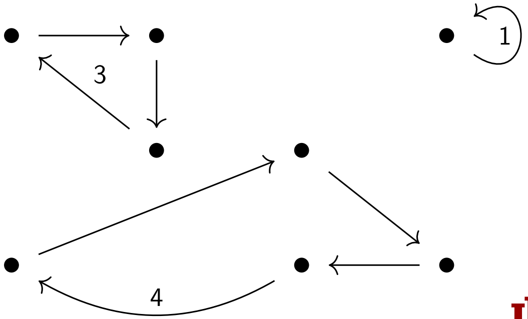
3. Record the cycle type.



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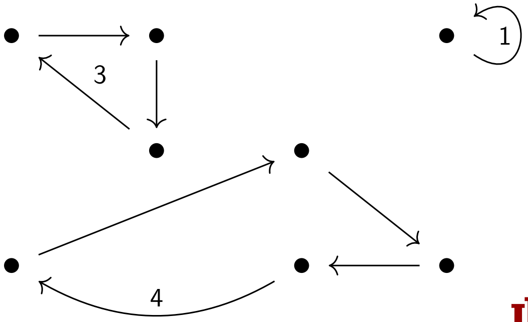
For this example, $\tau(f) = (4, 3, 1)$



Algebraic Structure

The Case of Finite Sets

P the set of all partitions inherits a natural semiring structure:

- $\tau(f) + \tau(g) := \tau(f \sqcup g)$
- $\tau(f) \cdot \tau(g) := \tau(f \times g)$
- $0 = ()$ and $1 = (1)$.
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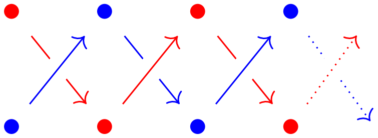

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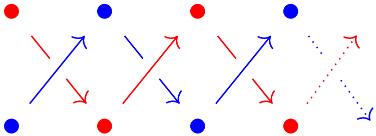

EG:

$$z_2 \cdot z_4 \ = \ 2z_4$$

Boredom Check

The Case of Finite Sets

1. Try $(z_3 + z_2) \cdot (z_5 + 2)$
2. What is $\tau(f)$ when $f(x) = 3x - x^3$ on the set $\{0, \pm 1, \pm 2\}$?

Finite Dynamics

Let $f\colon X \to X$, for some finite set X.

- $\tau(f)$ is the cycle type of the restriction of f to the largest subset on which f acts by a permutation.
- In terms of dynamics, $\tau(f)$ is a description of the *stable set* of the map f.

Familiar Invariants

Finite Dynamics

Let $F_n : P \to \mathbb{N}$ satisfy $F_n(z_m) = \delta_{n,m}$

- $\left(F_1 \circ \tau\right)$ is
- $\left(F_n \circ \tau\right)$ is
- $n \cdot \left(F_n \circ \tau\right)$ is
- $\sum_{d|n} d \cdot \left(F_d \circ \tau\right)$ is
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- $\sum_{d|n} d \cdot \left(F_d \circ \tau\right)$ is the fixed points of f^n.
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- $\sum_{n=1}^{\infty} n \cdot \left(F_n \circ \tau\right)$ is the cardinality of the stable set.

Counting Stable Sets

Finite Dynamics

Given $|X| = n$, how many stable sets are possible for the dynamical system (X, f)?

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Thinking of the system up to conjugacy, this means counting all possible partitions of the form $\tau(f)$:

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This is OEIS# A026905

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Applications to Group Theory

A G-representation (X, ρ) in a category $\mathcal{C}$ is a monoid homomorphism $\rho : G \to \mathrm{End}_{\mathcal{C}}(X)$.

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- Using the trace, each G-rep (V, ρ) in **Vec** produces a character:

$$\chi_\rho = \mathrm{tr} \circ \rho : G \to \mathrm{End}_{\mathbf{Vec}}(V) \to \mathbb{C}.$$

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- For any $\mathcal{C}$, we can use its universal trace τ_C to extend character theory to the category of G-reps in $\mathcal{C}$.

G-Sets

Applications to Group Theory

A G-representation (X, ρ) in **FinSet** is just a finite G-set.

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Applications to Group Theory

A G-representation (X, ρ) in **FinSet** is just a finite G-set.

- All G-sets decompose into orbits
- Each orbit is isomorphic to G/H, for some $H \leq G$.
- Each orbit encodes a character

Character Tables for S_3

Applications to Group Theory

	$[1]$	$[(12)]$	$[(123)]$
$\mathbb{1}$	1	1	1
σ	1	-1	1
V	2	0	-1

	$[(1)]$	$[(12)]$	$[(123)]$
S_3/S_3	1	1	1
S_3/C_3	2	z_2	2
S_3/C_2	3	$1+z_2$	z_3
$S_1/1$	6	$3z_2$	$2z_3$

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$$\mathcal{R}(S_3) \cong \frac{\mathbb{Z}[X, Y]}{(X^2 - 2x,\ Y^2 - XY - Y)}$$

Future Directions

- Other small/combinatorial categories
- Connections with Hochschild homology
- Making the construction functorial

Thank You

Boredom Update

1. Try $(z_3 + z_2) \cdot (z_5 + 2)$
2. What is $\tau(f)$ when $f(x) = 3x - x^3$ on the set $\{0, \pm 1, \pm 2\}$?

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Answer: $\tau(f) = 1 + z_2$