Morita Theory of Fusion (Higher) Categories

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Topics

Introduction

- Fusion 1-Categories
- Morita Theory
- Fusion 2-Categories
- Morita Theory of Fusion 2-Categories

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Fusion 1-Categories

Definition

A fusion category is a $\mathbb{C}$-linear, abelian, finite, semisimple, rigid monoidal category with simple unit object $\mathbb{1}$. (When $\mathbb{1}$ isn't simple it's multifusion).

Examples:

- Rep(G): the category of f.d. representations of a finite group G
- Vec$_G$: the category of f.d. G-graded vector spaces
- $\mathcal{C}(\mathfrak{g}, k)$: the quantum group category of a semisimple Lie alg. $\mathfrak{g}$ at level k.

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General Properties

Fusion 1-Categories

- All objects decompose as direct sums of simple objects.
- Fusion rules are described by the numbers $\{N_{X,Y}^{Z}\}_{X,Y,Z}$, where $X \otimes Y \cong \bigoplus_Z N_{X,Y}^{Z} Z$.
- Associators can be concretely described in terms of F-symbols (e.g. $6j$-symbols).
- They categorify rings, so many ring-theoretic ideas can be ported to the world of fusion categories.

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Modules at any Level

- 0-cats (vector spaces): $C \to \mathrm{End}(M)$ "M is a C module"
- 1-cats: $\mathcal{C} \to \mathrm{End}(\mathcal{M})$ "$\mathcal{M}$ is a $\mathcal{C}$ module category"
- 2-cats: $\mathfrak{C} \to \mathrm{End}(\mathfrak{M})$ "$\mathfrak{M}$ is a $\mathfrak{C}$ module 2-category"

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Morita Theory

Idea

Just as different bases can give different presentations of the same vector space, different algebras can give different presentations of the same category.

Definition

Two algebras A and B are said to be Morita equivalent and we'll write $A \overset{\text{Mor}}{\sim} B$ whenever

$$\mathbf{Mod}(A) \simeq \mathbf{Mod}(B)$$

The Following are Equivalent

Morita Theory

Let A and B be f.d. algebras.

- $\mathbf{Mod}(A) \simeq \mathbf{Mod}(B)$
- $\exists M, N$ such that $N \otimes_B M \cong A$, and $M \otimes_A N \cong B$
- $\exists$ a progenerator $M \in \mathbf{Mod}(B)$ with $A \cong \mathrm{End}_B(M)^{mop}$

Theorem [Morita 1957]

$A \overset{\mathrm{Mor}}{\sim} B$ implies $Z(A) \cong Z(B)$.

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The Following are Equivalent

Morita Theory

Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories.

- $\mathbf{Mod}(\mathcal{C}) \simeq \mathbf{Mod}(\mathcal{D})$
- $\exists \mathcal{M}, \mathcal{N}$ such that $\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{M} \cong \mathcal{C}$, and $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{D}$
- $\exists$ an separable $\mathcal{M} \in \mathbf{Mod}(\mathcal{D})$ with $\mathcal{C} \simeq \mathbf{End}_{\mathcal{D}}(\mathcal{M})^{mop}$

Theorem [Etingof-Nikshych-Ostrik 2011]

$\mathcal{C} \overset{\text{Mor}}{\sim} \mathcal{D}$ if and only if $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D})$.

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- $\exists$ an separable $\mathcal{M} \in \mathbf{Mod}(\mathcal{D})$ with $\mathcal{C} \simeq \mathbf{End}_{\mathcal{D}}(\mathcal{M})^{mop}$
  $\mathcal{M}$ is said to be an invertible $(\mathcal{D}, \mathcal{C})$-bimodule category.

Theorem [Etingof-Nikshych-Ostrik 2011]

$\mathcal{C} \overset{\text{Mor}}{\sim} \mathcal{D}$ if and only if $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D})$.

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A Group Theoretical Example

Morita Theory

There is a functor $F : \mathrm{Vec}_G \to \mathrm{Vec}$ that forgets the grading. This makes Vec a Vec_G module category by

$$X \triangleright V \quad := \quad F(X) \otimes V$$

It can then be computed that $\mathbf{End}_{\mathrm{Vec}_G}(\mathrm{Vec})^{mop} \simeq \mathrm{Rep}(G)$, and so

$$\mathrm{Vec}_G \overset{\mathrm{Mor}}{\sim} \mathrm{Rep}(G)$$

Slightly More Interesting Examples

Morita Theory

Whenever $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, each $\mathcal{C}_g$ is an invertible $(\mathcal{C}_e, \mathcal{C}_e)$-bimodule category.

- In TY categories $\langle m \rangle = \mathrm{Vec}$ is a rank 1 invertible $(\mathrm{Vec}_A, \mathrm{Vec}_A)$-bimodule category.
- $\mathcal{C} = \mathrm{Rep}\big(\mathrm{SL}_2(\mathbb{F}_3)\big)$ is $\mathbb{Z}/2$-graded, so $\mathcal{C}_1$ is an invertible bimodule category over $\mathcal{C}_0 = \mathrm{Rep}(A_4)$. Here $\mathrm{rk}(\mathcal{C}_0) = 4$ and $\mathrm{rk}(\mathcal{C}_1) = 3$.

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(2+1)D Topological Phases via String Nets

- In Levin-Wen models, ground states and excitations are described by $\mathcal{Z}(\mathcal{C})$ (the Drinfel'd center).
- By the Theorem[ENO'11], Morita equivalent fusion categories must produce the same phases.
- Kitaev and Kong developed a generalization of Levin-Wen where boundaries and (codim 1) defects are labelled by invertible bimodule categories.

(2+1)D Topological Phases via String Nets

- Defects labelled by bimodule categories.
- Invertibility of $\mathcal{M}$ implies bulk excitations can pass through without getting ‘stuck’

A. Kitaev, L. Kong

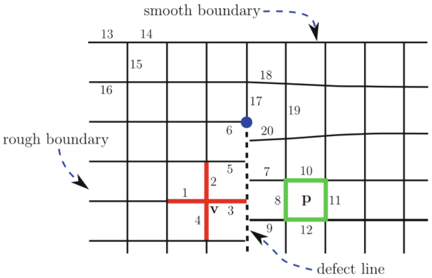

Fig. 1. Toric code with boundaries of two types and a defect line

Discussion

Fusion 2-Categories

Definition

A fusion 2-category is a finite semisimple rigid monoidal 2-category with simple monoidal unit.

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Theorem [Douglas-Reutter 2018]

Every finite semisimple 2-category is of the form $\mathbf{Mod}(\mathcal{C})$, with $\mathcal{C}$ a multifusion 1-category.

Properties

- Every object splits as a finite direct sum of simple objects
- A finite semisimple 2-category is locally a multifusion 1-category

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Examples

- $\mathbf{2Vect} = \mathbf{Mod}(\mathbf{Vect})$ the fusion 2-category of 2-vector spaces
- $\mathbf{Mod}(\mathcal{B})$ with $\mathcal{B}$ a braided fusion 1-category
- $\mathbf{2Rep}(G)$ the fusion 2-category of 2-representations of a finite group G
- $\mathbf{2Vect}_G$ the fusion 2-category of G-graded 2-vector spaces
- $\mathbf{2Vect}_G^\pi$ with π a 4-cocycle for G

Remark

In the above examples, G can be replaced by a finite 2-group $\mathcal{G}$.

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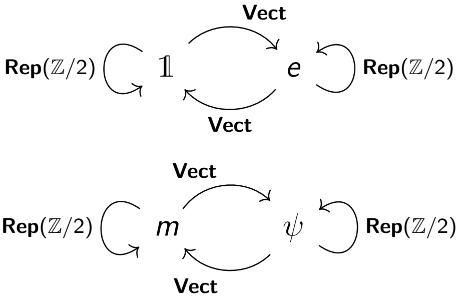
A Detailed Example

Take $\mathcal{G} = \mathbb{Z}/2\mathbb{Z}[2] \times \mathbb{Z}/2\mathbb{Z}[1]$ and $\mathbf{2Vect}_{\mathcal{G}} \simeq \mathbf{2Vect}_{\mathbb{Z}/2\mathbb{Z}} \boxtimes \mathbf{2Rep}(\mathbb{Z}/2\mathbb{Z})$.



$\Box$	$\mathbb{1}$	e	m	ψ
$\mathbb{1}$	$\mathbb{1}$	e	m	ψ
e	e	$2e$	ψ	2ψ
m	m	ψ	$\mathbb{1}$	e
ψ	ψ	2ψ	e	$2e$

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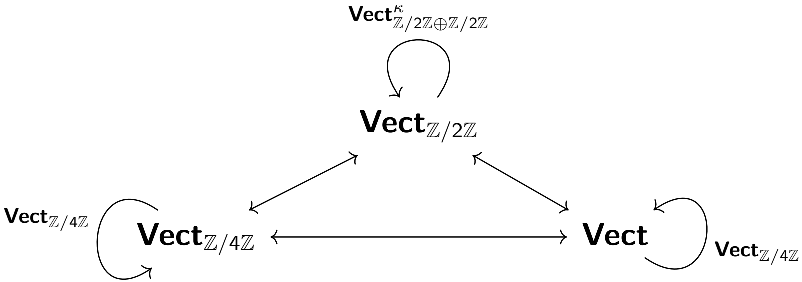
A Second Example

Consider $\mathbf{Mod}(\mathbf{Vect}_{\mathbb{Z}/4\mathbb{Z}})$.



Morita Theory of Fusion 2-Categories

Theorem [D. 2022]

Let $\mathfrak{C}$ be any fusion 2-category, and let $\mathfrak{M}$ be a separable $\mathfrak{C}$-module 2-category. Then, $\mathbf{End}_{\mathfrak{C}}(\mathfrak{M})$, the monoidal 2-category of $\mathfrak{C}$-module 2-endofunctors on $\mathfrak{M}$, is a multifusion 2-category.

Definition

Two fusion 2-categories $\mathfrak{C}$ and $\mathfrak{D}$ are Morita equivalent if there exists a separable $\mathfrak{C}$-module 2-category $\mathfrak{M}$, and an equivalence $\mathfrak{D} \simeq \mathbf{End}_{\mathfrak{C}}(\mathfrak{M})^{mop}$ of fusion 2-categories.

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Group Theoretic Examples

Fusion 2-Categories associated to Finite Groups

Let G be a finite group. Then, $\mathbf{2Vect}_G$ and $\mathbf{2Rep}(G)$ are Morita equivalent via $\mathbf{2Vect}$.

Fusion 2-Categories associated to Finite 2-Groups

Let $\mathcal{G}$ be a finite 2-group. Then, $\mathbf{2Vect}_{\mathcal{G}}$ and $\mathbf{2Rep}(\mathcal{G})$ are Morita equivalent via $\mathbf{2Vect}$. With $\pi_1(\mathcal{G}) = H$ and $\pi_2(\mathcal{G}) = A$, $\mathbf{2Vect}_{\mathcal{G}}$ is Morita equivalent to $\mathbf{2Vect}^{\omega}_{H \ltimes A}$, for a 4-cocycle ω.

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Examples arising from Braided Fusion 1-Categories I

Morita-trivial Fusion 2-Categories

A fusion 2-category is Morita equivalent to **2Vect** if and only if it is of the form $\mathbf{Mod}(\mathcal{Z}(\mathcal{C}))$ with $\mathcal{C}$ a fusion 1-category.

Invertible Fusion 2-Categories

- Invertible fusion 2-categories are of the form $\mathbf{Mod}(\mathcal{B})$ with $\mathcal{B}$ a non-degenerate braided fusion 1-category.
- Given $\mathcal{B}_1$ and $\mathcal{B}_2$ two non-degenerate braided fusion 1-categories, $\mathbf{Mod}(\mathcal{B}_1)$ and $\mathbf{Mod}(\mathcal{B}_2)$ are Morita equivalent if and only if $\mathcal{B}_1$ and $\mathcal{B}_2$ are Witt equivalent.

Examples arising from Braided Fusion 1-Categories II

Morita autoequivalences of $\mathbf{2Rep}(G) \simeq \mathbf{Mod}(\mathbf{Rep}(G))$

- Morita autoequivalences of $\mathbf{2Rep}(G)$ are parameterized by $H^3_{gr}(G;\mathbb{C}^\times)$.
- The corresponding separable module 2-categories are given by $\mathbf{Mod}(\mathbf{Vect}_G^\omega)$, the action is recorded by $\mathbf{Rep}(G) \hookrightarrow \mathcal{Z}(\mathbf{Vect}_G^\omega)$.

Discussion

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