A Curious Category

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Based on jt. work w/

Thibault Décoppet.

https://arxiv.org/abs/2412.15019

Slogan: OK yeah...

but what IS it?

⚠ Disclaimer: Everything is SemiSimple/Separable (fin. dim.)

Outline:

1) Background & discovery
2) Description of B
3) Interpretation & further directions

For me, an n-algebra is...

n	n-algebra (over $\mathbb{K}$)
0	f.d. semisimple alg $/\mathbb{K}$
1	fusion categories $/\mathbb{K}$
2	fusion 2 categories (compact semisimple $\otimes$ 2 categories)

Def: An n-algebra $\mathcal{C}$ is $\boxed{\text{(Morita) invertible}}$

if $\exists\, \mathcal{D}$ s.t.

$$\mathcal{C} \boxtimes \mathcal{D} \overset{\text{Morita}}{\sim} n\mathrm{Vec} = \mathbb{1}$$

Building invertible algebras

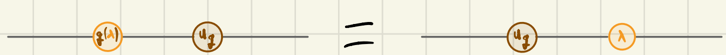
n = 0 : $\mathbb{L} \supseteq \mathbb{K}$, $G = \mathrm{Gal}(\mathbb{L}/\mathbb{K})$

$$[\varphi] \in H^2(G ; \mathbb{L}^\times)$$

$$\mathbb{L}^\varphi[G] = \mathrm{Span}_{\mathbb{L}}\{u_g \mid g \in G\}$$

$$u_g \cdot \lambda = g(\lambda) \cdot u_g$$

$$u_g \cdot u_h = \varphi(g,h) \cdot u_{gh}$$



These are central simple alg.
($\mathrm{Br}(\mathbb{K})$)

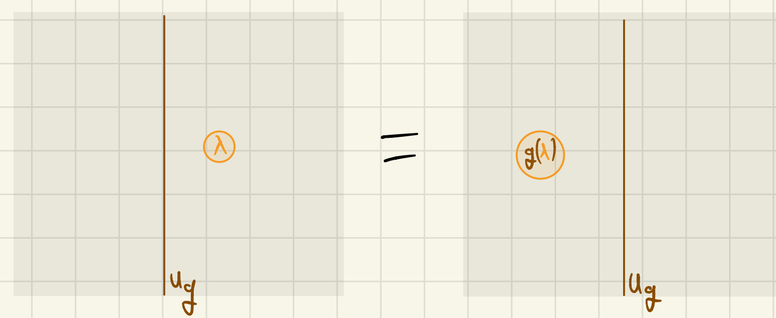
n = 1 : $\mathbb{L} \supseteq \mathbb{K}$, $G = \mathrm{Gal}(\mathbb{L}/\mathbb{K})$

$$[\omega] \in H^3(G ; \mathbb{L}^\times)$$

$$\mathrm{Vec}_{\mathbb{L}}^{\omega}(\mathrm{Gal}(\mathbb{L}/\mathbb{K}))$$

eg) $\mathbb{K} = \mathbb{C}(x,y,z)$

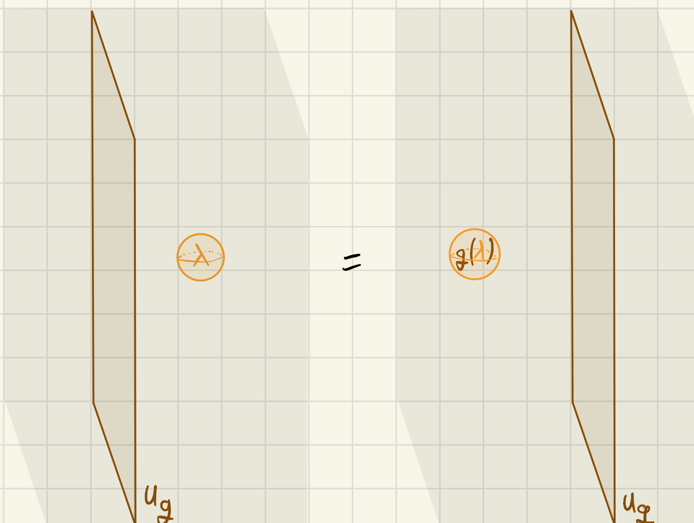
(Galois-nontriviality)



$\underline{n=2}$: $\mathbb{L} \supseteq \mathbb{K}$, $G = \mathrm{Gal}(\mathbb{L}/\mathbb{K})$

$$[\pi] \in H^4(G;\mathbb{L}^\times)$$

$$2\mathrm{Vec}_{\mathbb{L}}^{\pi}\left(\mathrm{Gal}(\mathbb{L}/\mathbb{K})\right)$$



Here $\overline{\mathbb{K}}$ ← The separable closure

Theorem: (Noether)

Invertible 0-algebras are classified by $H^2(\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K});\overline{\mathbb{K}}^\times)$.

Theorem: (S-Snyder) https://arxiv.org/abs/2407.02597

Invertible 1-algebras are classified by $H^3(\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K});\overline{\mathbb{K}}^\times)$.

Fact:

Invertible 2-algebras are $\boxed{\text{NOT!}}$ classified by $H^4(\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K});\overline{\mathbb{K}}^\times)$.

~~Building~~ More invertible algebras

Some facts:

(i) If $\mathcal{B}$ is a braided 1-algebra, then $\mathrm{Mod}(\mathcal{B})$ is a 2-algebra

(ii) $Z(\mathcal{C}) \overset{\text{Witt}}{\sim} \mathrm{Vec}$

(iii) $\mathcal{B}_1 \overset{\text{Witt}}{\sim} \mathcal{B}_2 \Rightarrow \mathrm{Mod}(\mathcal{B}_1) \overset{\text{Morita}}{\sim} \mathrm{Mod}(\mathcal{B}_2)$

(iv) $\mathcal{B}$ nondegenerately braided $\Rightarrow Z(\mathcal{B}) \simeq \mathcal{B} \boxtimes \mathcal{B}^{rev}$

Fact: If $\mathcal{B}$ is nondegenerately braided, then $\mathrm{Mod}(\mathcal{B})$ is invertible.

~~proof~~:

$$\mathrm{Mod}(\mathcal{B}) \boxtimes \mathrm{Mod}(\mathcal{B}^{rev})$$
$$\simeq \mathrm{Mod}(\mathcal{B} \boxtimes \mathcal{B}^{rev})$$
$$\simeq \mathrm{Mod}(Z(\mathcal{B}))$$
$$\overset{\text{Morita}}{\sim} \mathrm{Mod}(\mathrm{Vec}) = 2\mathrm{Vec}$$

$$\rightsquigarrow \mathrm{Witt}(\mathrm{Vec}_{\mathbb{C}})$$

Theorem: (S-Décoppet)

Let $\mathbb{k}$ be any field & $\mathcal{C}$ any 2-algebra.
There exists a braided 1-algebra $\mathcal{B}$ s.t.

$$\mathcal{C} \overset{\text{Morita}}{\sim} \mathrm{Mod}(\mathcal{B})$$

Corollary:

Every (Morita equivalence class of) invertible 2-algebra is represented by $\mathrm{Mod}(\mathcal{B})$ for some braided 1-algebra $\mathcal{B}$.

Galois cohomology of $\mathbb{R}$

$$\overline{\mathbb{R}} = \mathbb{C} \quad , \quad \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$$

$$H^n(\mathbb{Z}/2\mathbb{Z} ; \mathbb{C}^\times_{\mathrm{Gal}}) \cong \begin{cases} \mathbb{R}^\times & n = 0 \\ 0 & n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & n \text{ even} \end{cases}$$

$H^2(\mathbb{Z}/2\mathbb{Z} ; \mathbb{C}^\times_{\mathrm{Gal}})$	$H^4(\mathbb{Z}/2\mathbb{Z} ; \mathbb{C}^\times_{\mathrm{Gal}})$
$\mathbb{Z}/2\mathbb{Z} \cong \langle \mathbb{H} \rangle$	$\mathbb{Z}/2\mathbb{Z} \cong \langle ??? \rangle$

Our technique for constructing $\mathcal{B}$ w/

$$\mathrm{Mod}(\mathcal{B}) \simeq 2\mathrm{Vec}_{\mathbb{C}}^{\pi}(\mathrm{Gal}(\mathbb{C}/\mathbb{R}))$$

Shows that

Toric Code

$$\mathcal{B} \boxtimes_{\mathbb{R}} \mathrm{Vec}_{\mathbb{C}} \simeq \mathcal{Z}(\mathrm{Vec}_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z}))$$

So...

(1) $0 \neq [\pi] \in H^4(\mathbb{Z}/2\mathbb{Z} ; \mathbb{C}^{\times}_{\mathrm{Gal}})$

$\Rightarrow$

(2) $[\mathcal{B}] \in \mathrm{Ker}\left(\mathrm{Witt}(\mathrm{Vec}_{\mathbb{R}}) \to \mathrm{Witt}(\mathrm{Vec}_{\mathbb{C}})\right)$

Strategy:

Step 1: Use Galois descent of Etingof-Gelaki to find all the real forms of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z}))$

Step 2: Figure out which form is not a center.

(This is tedious, but it works!)

The action & the fixed points

$$Z(\mathrm{Vec}_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z})) = \left\langle \begin{matrix} \overset{\text{Fixed}}{\mathbb{1}} & & e \\ & m \xleftrightarrow{\text{swapped}} e & \\ m & & em^{\text{Fixed}} \end{matrix} \right\rangle_{\oplus}$$

"outer" $\qquad \beta_{e^i m^j, e^k m^l} = (-1)^{il} \cdot (\text{swap})$

$$(T, J) \in \mathrm{Aut}^{br}_{\mathbb{R}}(Z(\mathrm{Vec}_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z})))$$

$$T(\lambda \cdot id_{\mathbb{1}}) = \bar{\lambda} \cdot id_{\mathbb{1}}$$

$$T(e) = m$$

$$T(m) = e$$

"inner" $\qquad J_{e^i m^j, e^k m^l} = (-1)^{jk} \cdot id$

$$\begin{tikzcd}
T(e^i m^j) \otimes T(e^k m^l) \arrow[r, "J = (-1)^{jk}"] \arrow[d, "(-1)^{jk} = \beta_T"'] & T(e^i m^j \otimes e^k m^l) \arrow[d, "T\beta = (-1)^{il}"] \\
T(e^k m^l) \otimes T(e^i m^j) \arrow[r, "J = (-1)^{il}"'] & T(e^i m^j \otimes e^k m^l)
\end{tikzcd}$$

$$(T, J)^2 = (id, \ J^{(2)} = (-1)^{il + jk})$$

$$\Big\Downarrow \ \mu_{e^i m^j} = (-1)^{ij}$$

$$(id, \ id)$$

Simples in $\mathcal{B} = Z(\mathrm{Vec}_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}}$

$$\left(\mathbb{1},\ T(\mathbb{1}) \xrightarrow{id} \mathbb{1}\right) = \mathbb{1}_{\mathcal{B}}$$

$$\mathrm{End}(\mathbb{1}) = \mathbb{R}$$

$$\left(e\oplus m,\ T(e\oplus m) \longrightarrow e\oplus m\right) = X$$

$$\mathrm{End}(X) \cong \mathbb{C}$$

$$T(em) \xrightarrow[u]{\cong} em$$

but this is not coherent!

$$\begin{tikzcd}
T^2(em) \arrow[r, "Tu"] \arrow[d, "-id = \mu_{em}"'] & T(em) \arrow[d, "u"] \\
em \arrow[r, "id"'] & em
\end{tikzcd}$$

If $u = \lambda \cdot id$, then

$u \circ Tu = |\lambda|^2 \cdot id$

$\neq -id$

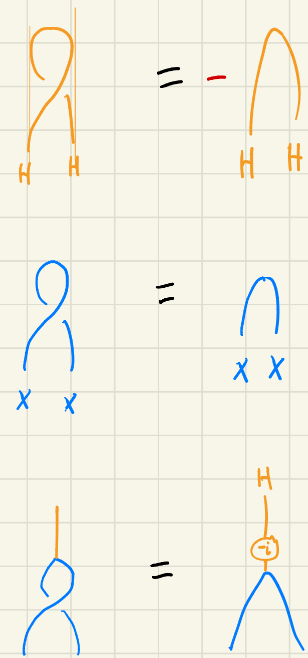
$$\left(em^{\oplus 2},\ T(em^{\oplus 2}) \xrightarrow{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} em^{\oplus 2}\right) = H$$

$$\mathrm{End}(H) \cong \mathbb{H}$$

Fusion Rules

	$\mathbb{R}$	$\mathbb{H}$	$\mathbb{C}$
	$\mathbb{1}$	H	X
$\mathbb{1}$	$\mathbb{1}$	H	X
H		$4 \cdot \mathbb{1}$	$2 \cdot X$
X			$2\,\mathbb{1} \oplus H$

Some braiding info



Some categorified fields

(This bit comes from Theo Johnson-Freyd's https://arxiv.org/abs/1507.06297)

$$\begin{tikzcd}
 & s\mathrm{Vec}_{\mathbb{C}} = \langle \mathbb{1}, f \rangle_\otimes \arrow[dl, dash, "B\mathbb{Z}/2\mathbb{Z}"'] \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \arrow[dr, dash, "\mathbb{Z}/2\mathbb{Z}"] & \\
\langle \mathbb{1} \rangle_\otimes = \mathrm{Vec}_{\mathbb{C}} \arrow[dr, dash, "\mathbb{Z}/2\mathbb{Z}"'] & s\mathrm{Vec}_{\mathbb{H}} = \langle \mathbb{1}, f \rangle_\otimes \arrow[d, dash, "B\mathbb{Z}/2\mathbb{Z}"] & s\mathrm{Vec}_{\mathbb{R}} = \langle \mathbb{1}, f \rangle_\otimes \arrow[dl, dash, "B\mathbb{Z}/2\mathbb{Z}"] \\
 & \langle \mathbb{1} \rangle_\otimes = \mathrm{Vec}_{\mathbb{R}} &
\end{tikzcd}$$

$$s\mathrm{Vec}_{\mathbb{H}} = \underbrace{\langle \mathbb{1}, H \rangle_\otimes}_{\text{dim } 2} \subseteq \underbrace{\langle \mathbb{1}, H, X \rangle_\otimes}_{\text{dim } 4} = \mathcal{B}$$

A min nondegenerate extension!

$$\langle \mathrm{Vec}_{\mathbb{C}} \rangle \subseteq \mathrm{Pic}(s\mathrm{Vec}_{\mathbb{H}})$$

Minimal nondegenerate extensions?

$$\begin{tikzcd}
\mathrm{Mext}(s\mathrm{Vec}_{\mathbb{C}}) \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Witt}(\mathrm{Vec}_{\mathbb{C}}) \arrow[r, two heads] & \mathrm{Witt}(s\mathrm{Vec}_{\mathbb{C}}) \\
\mathbb{Z}/16\mathbb{Z} = \langle d_{sig} \rangle & &
\end{tikzcd}$$

Not surjective!

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[r, hook] \arrow[d, phantom, "=" rotate=90] & \mathrm{Witt}(\mathrm{Vec}_{\mathbb{R}}) \arrow[r] & \mathrm{Witt}(\mathrm{Vec}_{\mathbb{C}}) \\
\langle \mathcal{B} \rangle & &
\end{tikzcd}$$

Physical interpretation

$(T, J) \in \mathrm{Aut}^{br}_{\mathbb{R}}(Z(\mathrm{Vec}_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z})))$

acts by complex conjugation

$\rightsquigarrow$ Time Reversal

The fact that $\mathcal{B} \neq Z(\mathcal{C})$ for any $\mathcal{C}$ means that $\mathcal{B}$ does not admit a gapped boundary.

Since $\mathcal{B} \boxtimes_{\mathbb{R}} \mathrm{Vec}_{\mathbb{C}}$ is a center, it does have a gapped boundary theory . . .

. . .

So it's a

(Time Reversal)

Symmetry

Protected

Topological

Phase of matter

Thanks!